Math 856 Homework 3

Starred (*) problems to be handed in Thursday, October 5

(*) 14: Show that if M, N are smooth manifolds, M is connected, and $f : M \to N$ is a smooth map with $f_* : T_aM \to T_{f(a)}N$ equal to the zero map for all $a \in M$, then f is the constant function. (Hint: show that $f^{-1}(\{f(a)\})$ is open! And beat the problem over the head with some calculus...)

(*) 15: For $a \in M$, let $\mathcal{F}_a \subseteq C^\infty(M)$ denote the smooth functions satisfying $f(a) = 0$. and let $L : \mathcal{F}_a \to \mathbb{R}$ be a linear operator satisfying $L(fg) = 0$ for all $f, g \in \mathcal{F}_a$. Show that there is a unique derivation $X \in T_aM$ satisfying $X|_{\mathcal{F}_a} = L$.

(N.B. This provides still another characterization of tangent vectors, as the vector space of linear maps $X : \mathcal{F}_a/W \to \mathbb{R}$, where $W = \mathcal{F}_a^2$ = the ideal generated by products fg for $f, g \in \mathcal{F}_a$.)

16: The tangent space for a manifold M with boundary is defined in exactly the same way as for a manifold; the derivations at a point in ∂M are allowed to point "in all the directions" of $\mathbb{R}^n$.

We say that a tangent vector $X \in T_aM$ for $a \in \partial M$ "points inward" if in some set of local coordinates $h = (x^1, \dots, x^n)$ we have $X = \sum_i v^i \partial/\partial x^i$ with $v^n > 0$. (Here h maps to the upper half-space, where $x^n \geq 0$.) Show that the notion of "pointing inward" is independent of coordinate chart.

17: Show that the C^∞ manifolds $T(M \times N)$ and $TM \times TN$ are diffeomorphic.

18: Show that if M^n is a compact smooth manifold that admits a smooth embedding into $\mathbb{R}^{n+1}$, then $M^n \times S^k$ admits a smooth embedding into $\mathbb{R}^{n+k+1}$.

Show that for any $n_1, \dots, n_k \geq 1$ and $N = \sum_i n_i$, $S^{n_1} \times \cdots \times S^{n_k}$ admits a smooth embedding into $\mathbb{R}^{N+1}$.