Math 871 Exam 2 Topics, PART 1 (the point-set part)

Compactness. The topologcal property behind the Extreme Value Theorem

Same idea: what would make EVT fail? $f : X \to \mathbb{R}$ (Think maximum.) Either f has no upper bound, or its image has a (least) upper bound that is not achieved. Either, taking inverse images, leads to a collection of nested open sets whose union is X, but none of them is X. This leads to:

An *open covering* of a topological spac X is a collection of open sets $\mathcal{U}_\alpha$ whose union is X.

$(X, \mathcal{T})$ is *compact* (cpct) if every open covering $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ has a *finite subcover(ing)*: a finite collection $J \subseteq I$ so that $\{\mathcal{U}_\beta\}_{\beta\in J}$ is also an open covering of X.

Compact subset $A \subseteq X$: $(A, \mathcal{T}_A)$ is a cpct space.

Topologist's EVT: If $f : (X, \mathcal{T}) \to (\mathbb{R}, \text{usual})$ is cts and X is cpct, then there are $a, b \in X$ so that $f(a) \leq f(x) \leq f(b)$ for all $x \in X$.

'Real' Topologists EVT: if $f : (X\mathcal{T}) \to (Y, \mathcal{T}')$ is cts and X is cpct, then $f(X) \subseteq Y$ is a cpct subset of Y.

A closed subset of a compact set is compact.

A compact subset of a Hausdorff space is closed.

Consequence:

If $f : X \to Y$ is a cts bijection, and X is cpct while Y is Hausdorff, then f is a homeomorphism.

This is generally our favorite way, for example, to identify a space as a quotient of another space; $f : X \to Y$ cts and surjective, with X cpct and Y Hausdorff, then the quotient space $X/\sim$ where $x \sim y$ iff $f(x) = f(y)$, is homeomorphic to Y.

X and Y cpct, then $X \times Y$ is cpct.

If X_α are compact for all α, then $\prod X_\alpha$ is compact, using the product topology. [This is a deep result...]

A finite union of compact subsets is compact.

Closed set formulation: take complements! A collecction A_α of subsets of X has the *finite intersection property* (FIP) if the intersection of any finite number of the A_α is non-empty.

X is cpct $\Leftrightarrow$ for any collection $\{C_\alpha\}$ of closed subsets of X that has the FIP, we have $\bigcup_\alpha C_\alpha \neq \emptyset$.

This is often treated as a method for locating 'interesting' points: in a cpct space X if you can satisfy any finite number of a collection of 'closed' conditions (the pts satisfying each condition form a closed subset) then you can simultaneously satisfy all of them.

Alternate notions: In any cpct space, every infinite subset has a limit point.

$(X, \mathcal{T})$ is *limit point compact* if every infinite subset of X has a limit point.

limit pt cpctness does not imply cpctness. Limit pt cpctness is not preserved under cts image. For *metrizable* spaces X (those whose open sets can be defined by a metric), X is cpctness $\Leftrightarrow$ X is limit pt cpct.

Nets:

A *sequence* in X is a function $f : \mathbb{Z}_+ \to X$; $f(n) = x_n \in X$. A sequence converges $x_n \to x$ if for all $x \in U \in X$, $U \in \mathcal{T}$, there is an $N \in \mathbb{Z}_+$ so that $n \geq N \Rightarrow x_n \in U$.

If $f : X \to Y$ is cts and $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y. But in like in analysis/$\mathbb{R}^n$, the converse is not true; convergence of sequences is not good enough to imply continuity.

A *net* in X is a collection of points indexed by a *directed set* $(D, \geq)$ [$a \geq b$ and $b \geq c$ implies $a \geq c$], where for all $a, b \in D$ there is a $d \in D$ so that $d \geq a$, $d \geq b$

Model: all $U \in \mathcal{T}$ with $x \in U$, where $U \geq V$ means $V \subseteq U$ (reverse inclusion)

A net $\{x_i\}_{i \in D}$ in X *converges* if there is an $x \in X$ so that $x \in U \in \mathcal{T}$ implies there is a $d \in D$ so that $i \geq d \Rightarrow x_i \in U$. [Write $x_i \to x$]

If $f : X \to Y$ is cts and $x_i \to x$ in X, then $f(x_i)$ (is a net in Y, with the same index set and) converges to $f(x)$.

But the converse is now true: if the image under f of every convergent net is a convergent net, then f is continuous.

Also: a space X is cpct $\Leftrightarrow$ Every net in X has a convergent *subnet* [a subnet must be cofinal: it is defined by a subset $E \subseteq D$ so that for all $d \in D$ there is some $e \in E$ with $e \geq d$.] The corresponding statement for sequences/subsequences is false (but they do characterize cpctness for metrizabe spaces).

Countability and separation properties.

Idea: other useful properties that $\mathbb{R}$ has that we might like $(X, \mathcal{T})$ to have.

$(X, \mathcal{T})$ is *separable* if there is a countable subset $A \subseteq X$ with $\overline{A} = X$ [A is dense in X].

$(X, \mathcal{T})$ is *second countable* if $\mathcal{T} = \mathcal{T}(\mathcal{B})$ for some countable basis $\mathcal{B} \subseteq \mathcal{T}$.

$(X, \mathcal{T})$ is *first countable* if for every $x \in X$ there are $\{U_n\}_{n \in \mathbb{Z}_+} \subseteq \mathcal{T}$ so that $x \in U \in \mathcal{T}$ implies $x \in U_n \subseteq U$ for some n. [Each point has a countable *neighborhood basis*.]

$(X, \mathcal{T})$ metrizable implies first ctble.

$(X, \mathcal{T})$ second ctble $\Rightarrow$ $(X, \mathcal{T})$ separable, $(X, \mathcal{T})$ first ctble.

In general, separable and first ctble do <u>not</u> imply second ctble.

Subspace of first ctble is first ctble; subspace of second ctble is second ctble.

Separation properties: T_1 (points are closed) and T_2 = Hausdorff we have met already.

A space $(X, \mathcal{T})$ is T_3 if for every $C \subseteq X$ closed and $x \in X$ with $x \notin C$, there are $U, V \in \mathcal{T}$ so that $x \in U$, $C \subseteq V$, and $U \cap V = \emptyset$.

A space $(X, \mathcal{T})$ is T_4 if for every $C, D \subseteq X$ closed with $C \cap D = \emptyset$, there are $U, V \in \mathcal{T}$ so that $C \subseteq U$, $D \subseteq V$, and $U \cap V = \emptyset$.

$(X, \mathcal{T})$ is *regular* if it is T_1 and T_3; $(X, \mathcal{T})$ is *normal* if it is T_1 and T_4.

normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff $\Rightarrow$ T_1

$(X, \mathcal{T})$ metrizable implies that X is normal.

$(X, \mathcal{T})$ compact and Hausdorff implies that $(X, \mathcal{T})$ is normal.

$(X, \mathcal{T})$ regular and second ctble implies that $(X, \mathcal{T})$ is normal.

Alternate forms:

$(X, \mathcal{T})$ is regular $\Leftrightarrow$ X is T_1 and whenever $x \in U \in \mathcal{T}$, there is a $V \in \mathcal{T}$ with $x \in V \subseteq \overline{V} \subseteq U$.

$(X, \mathcal{T})$ is normal $\Leftrightarrow$ X is T_1 and whenever $C \subseteq U \in \mathcal{T}$ with C closed, there is a $V \in \mathcal{T}$ with $C \subseteq V \subseteq \overline{V} \subseteq U$.

The cartesian product of regular spaces is regular; this is <u>not</u> true for normal spaces (and normality).

$(X, \mathcal{T})$ is *metrizable*, then X is normal and first countable; these are necessary. What is <u>sufficient</u>?

Urysohn Metrization Theorem: If $(X, \mathcal{T})$ regular and second countable, then [$(X, \mathcal{T})$ is normal and] $(X, \mathcal{T})$ is metrizable.

The idea: build an embedding $X \hookrightarrow \prod_{n \in \mathbb{Z}_+} \mathbb{R}$, so X is a subspace of a metrizable space, hence metrizable. Key ingredient:

Urysohn's Lemma: If $(X, \mathcal{T})$ is normal then for $A, B \subseteq X$ closed with $A \cap B = \emptyset$, there is a cts $f : X \to [0, 1]$ so that $f|_A = 0$ and $f|_B = 1$.

So second ctbility, in addition to normality, is sufficient; but there are non-second-countable metric spaces (uncountable set, discrete topology!).

Smirnov metrizability: $(X, \mathcal{T})$ is metrizable $\Leftrightarrow$ X is Hausdorff, paracompact, and locally metrizable.

Nagata-Smirnov metrizability: $(X, \mathcal{T})$ is metrizable $\Leftrightarrow$ X is regular and has a σ-locally-finite basis for the topology.

Paracompact = every open covering has a locally finite refinement. Refinement = an open covering each of which is a subset of one of the original covering. Locally finite = every point as a neighborhood meeting only finitely many of the sets. σ-locally-finite = is a countable union of locally finite collections.

Local properties: nearly every one of the properties we have studied has a "local" version (essentially, it hold for some open subset of a point).

Locally connected: Given $x \in U \in \mathcal{T}$, there is a $V \in \mathcal{T}$ with $x \in V \subseteq U$ and V is connected.

Locally path connected: Given $x \in U \in \mathcal{T}$, there is a $V \in \mathcal{T}$ with $x \in V \subseteq U$ and V is path connected.

Locally compact: Given $x \in X$, there is a $U \in \mathcal{T}$ and $C \subseteq X$ compact, so that $x \in U \subseteq C$.

All of these are topological properties, and they typically allow us to leverage the useful properties that follow from connectedness/path connectedness/compactness to more general settings. [Think: $\mathbb{R}$ is not cpct, but it is locally cpct.]

Homotopy Theory.