

Math 871 Problem Set 3

Starred (*) problems are due Thursday, Sept. 18.

13. [A useful openness test.] If (X, \mathcal{T}) is a topological space, and $A \subseteq X$, show that $A \in \mathcal{T}$ if and only if

for all $x \in A$, there is a $U \in \mathcal{T}$ so that $x \in U \subseteq A$

14. Show that if $f, g : X \rightarrow Y$ are two functions from the topological space (X, \mathcal{T}) to Y , then the finest topology on Y which makes both functions continuous is the intersection of the finest topologies making each function alone continuous.

- (*) 15. Show that if \mathcal{B} and \mathcal{B}' are both bases for topologies on X , then so is $\mathcal{B} \cap \mathcal{B}'$, but $\mathcal{B} \cup \mathcal{B}'$ need not be.

16. [Munkres, p.83, Problem #5] Show that if \mathcal{B} is a basis for a topology on X , then $\mathcal{T}(\mathcal{B})$ is the intersection of all topologies that contain \mathcal{B} . Show that the analogous statement is true for the topology generated by a subbasis.

- (*) 17. Show that, for any set X , the set

$$\mathcal{B} = \{B \subseteq X : X \setminus B \text{ is infinite}\} \cup \{X\}$$

is a basis for a topology on X . What (familiar!) topolog(ies) does it generate?

18. [Munkres, p.83, Problem #8] (a) Show that the collection $\mathcal{B} = \{(a, b) : a, b \in \mathbb{Q}\}$ is a basis, which generates the 'usual' topology on \mathbb{R} .

(b) Show, by contrast, that the collection $\mathcal{B}' = \{[a, b) : a, b \in \mathbb{Q}\}$ is a basis, but the topology it generates is strictly coarser than the 'lower limit' topology \mathcal{T}_ℓ on \mathbb{R} .

- (*) 19. Show that for any pair of topologies \mathcal{T} and \mathcal{T}' on \mathbb{R} , the product topology on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ that they generate **cannot** be equal to the finite complement topology on \mathbb{R}^2 .

[What does the complement of a box look like?]

20. [Munkres, p.92, Problem #5] If $\mathcal{T} \subseteq \mathcal{T}'$ are topologies on the set X and $\mathcal{O} \subseteq \mathcal{O}'$ are topologies on the set Y , show that the product topology $\mathcal{T} \times \mathcal{O}$ on $X \times Y$ is coarser than the topology $\mathcal{T}' \times \mathcal{O}'$. Is the converse result true [i.e., product topology coarser implies that the topologies on each factor are coarser]?