Math 871 Problem Set 4

Starred (*) problems are due Thursday, Sept. 25.

21. Theorem 15.1 of the text shows that if $(X, \mathcal{T})$ and $Y, \mathcal{T}'$ are topological spaces with bases $\mathcal{T} = \mathcal{T}(\mathcal{B})$ and $\mathcal{T}' = \mathcal{T}(\mathcal{B}')$, then $\mathcal{B} \times \mathcal{B}' = \{B \times B' \ : \ B \in \mathcal{B}, B' \in \mathcal{B}'\}$ is a basis for the product topology on $X \times Y$. [You should prove this for yourself!]

    Show, by contrast, that, for an infinite Cartesian product, one natural generalization of this is (typically) <u>false</u>; if $\mathcal{T}_\alpha = \mathcal{T}(\mathcal{B}_\alpha)$ for all α, the sets

    $$\prod_\alpha(\mathcal{B}_\alpha) = \{\prod_\alpha B_\alpha \ : \ B_\alpha \in \mathcal{B}_\alpha \text{ for all } \alpha\}$$

    is a basis, but for the <u>box</u> topology on $\prod_\alpha X_\alpha$, not (typically) the product topology. What would the 'correct' basis be, to generate the product topology?

(*) 22. [Munkres, p.112, Problem #10] Show that if A, B, X, Y are topological spaces, and $f : A \to X$ and $g : B \to Y$ are both continuous functions, then the function $h : A \times B \to X \times Y$, given by $h(a, b) = (f(a), g(b))$, is also continuous (using the product topologies on domain and codomain).

(*) 23. [Munkres, p.91, Problem #1] Show that if $(X, \mathcal{T})$ is a topological space, $A \subseteq X$ is given the subspace topology $\mathcal{T}_A$ it inherits from X, and $B \subseteq A$, then the subspace topology $\mathcal{T}_{B,A}$ that B inherits from A is the same as the subspace topology $\mathcal{T}_{B,X}$ that it inherits as a subset of X.

24. Show that if $\mathcal{T} \subseteq \mathcal{T}'$ are topologies on the set X, and $A \subseteq X$, then the subspace topology on A induced by $\mathcal{T}$ is coarser than the subspace topology induced by $\mathcal{T}'$. Find examples of topologies $\mathcal{T} \subsetneq \mathcal{T}'$ on $\mathbb{R}$ so that the topologies that they induce on $[0, 1] \subseteq \mathbb{R}$ are the <u>same</u>.

25. [Munkres, p.118m, Problem #2] Show that if $(X_\alpha, \mathcal{T}_\alpha$ are topological spaces and $A_\alpha \subseteq X_\alpha$ are given the subspace topologies for each α, and $\prod_\alpha X_\alpha$ is given the product topology $\mathcal{T}$, then the subspace topology on $\prod_\alpha A_\alpha$ induced by $\mathcal{T}$ is the product topology on $\prod_\alpha A_\alpha$. Is the analogous statement true if we use the box topologies?

(*) 26. Giving $\mathbb{R}$ the usual (metric) topology, show that if $f : (X, \mathcal{T}) \to \mathbb{R}$ and $g : (X, \mathcal{T}) \to \mathbb{R}$ are both continuous, then the functions $m, M : (X, \mathcal{T}) \to \mathbb{R}$, given by

    $$m(x) = \min\{f(x), g(x)\} \qquad\qquad M(x) = \max\{f(x), g(x)\}$$

    are also both continuous.

    [Hint: using the subbasis $\mathcal{S} = \{(a, \infty) \ : \ a \in \mathbb{R}\} \cup \{(-\infty, a) \ : \ a \in \mathbb{R}\}$ cuts down on your work...]

27. (a) Show that $\{(x, y) \in \mathbb{R}^2 \ : \ x + y > a\} = \bigcup_c (c, \infty) \times (a - c, \infty)$ and $\{(x, y) \in \mathbb{R}^2 \ : \ x + y < a\} = \bigcup_c (-\infty, c) \times (-\infty, a - c)$ (so both are open sets).

    (b) Show that the function $A : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (with the usual (product) toplogies) given by $A(x, y) = x + y$ is continuous. [Part (a) helps... see previous hint!]

    (c) Use a previous problem (!) to conclude that if $f, g : (X, \mathcal{T}) \to (\mathbb{R}, \text{usual})$ are both continuous , then the function $h : (X, \mathcal{T}) \to (\mathbb{R}, \text{usual})$ given by $h(x) = f(x) + g(x)$ is also continuous.