Math 871 Problem Set 3

Starred (*) problems are due Thursday, Sept. 15.

(*) 16. [A useful openness test.] If $(X, \mathcal{T})$ is a topological space, and $A \subseteq X$, show that $A \in \mathcal{T}$ if and only if

$$\text{for all } x \in A, \text{ there is a } U \in \mathcal{T} \text{ so that } x \in U \subseteq A$$

17. Show that if $f, g : X \to Y$ are two functions from the topological space $(X, \mathcal{T})$ to Y, then the finest topology on Y which makes both functions continuous is the intersection of the finest topologies making each function alone continuous.

18. Show that if $\mathcal{B}$ and $\mathcal{B}'$ are both bases for topologies on X, then the set
$$\mathcal{B}'' = \{B \cap B' : B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\}$$
is also a basis for a topology on X , and that the topology it generates is the coarsest topology on X containing both $\mathcal{B}$ and $\mathcal{B}'$.

(*) 19. [Munkres, p.83, Problem #5] Show that if $\mathcal{B}$ is a basis for a topology on X, then $\mathcal{T}(\mathcal{B})$ is the intersection of all topologies that contain $\mathcal{B}$. Show that the analogous statement is true for the topology generated by a subbasis.

20. Show that, for any set X, the set
$$\mathcal{B} = \{B \subseteq X : X \setminus B \text{ is infinite } \} \cup \{X\}$$
is a basis for a topology on X. What (familiar!) topolog(ies) does it generate?

21. [Munkres, p.83, Problem #8] (a) Show that the collection $\mathcal{B} = \{(a, b) : a, b \in \mathbb{Q}\}$ is a basis, which generates the 'usual' topology on $\mathbb{R}$.

(b) Show, by contrast, that the collection $\mathcal{B}' = \{[a, b) : a, b \in \mathbb{Q}\}$ is a basis, but the topology it generates is strictly coarser than the 'lower limit' topology $\mathcal{T}_\ell$ on $\mathbb{R}$.

(*) 22. Show that $\mathcal{B} = \{(a, \infty) \times (b, \infty) : a, b \in \mathbb{R}\}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, which is coarser than the usual Euclidean topology on $\mathbb{R}^2$. Show that $\mathcal{B}' = \{[a, \infty) \times [b, \infty) : a, b \in \mathbb{R}\}$ is a basis for a topology $\mathcal{T}'$ which is strictly finer than $\mathcal{T}$, and not comparable to the usual Euclidean topology.

23. [Munkres, p.92, Problem #5] If $\mathcal{T} \subseteq \mathcal{T}'$ are topologies on the set X and $\mathcal{O} \subseteq \mathcal{O}'$ are topologies on the set Y, show that the product topology $\mathcal{T} \times \mathcal{O}$ on $X \times Y$ is coarser than the topology $\mathcal{T}' \times \mathcal{O}'$. Is the converse result true [i.e., product topology coarser implies that the topologies on each factor are coarser]?

24. Show that if (X, d) and (Y, d') are metric spaces, then the product topology on $X \times Y$ is also a metric topology. [There are lots of (correct) choices of metric on $X \times Y$; you can take your cue from $\mathbb{R}^2$.]