Math 871 Topology I, Fall 2020

Exam 2

Instructions: Officially, this exam takes place at our scheduled final exam time: Tuesday, November 24, 1:00pm-3:00pm. You may, however, prepare your solutions in advance of this exam time, and, if you feel that you have complete solutions, you may turn them in to the instructor before the official exam time. You may either email your solutions to the instructor or use the file upload link provided from the course Canvas Home page. You should treat this as a self-timed exam, and limit yourself to six (6) hours of active work on the exam. You may partition that time in any way that best fits your schedule. [If you choose to (La)TeX your solutions, you do not need to include your typesetting time in the above limit; handwritten solutions are also (always!) fine, and may be preferable in light of the other things that you probably need to be doing in the coming weeks...]

In preparing your answers, you may consult our textbooks, your course notes, your solutions to the problem sets, the instructor's solutions to the problem sets, and the instructor's course notes on Canvas. You should not consult any other resource (to the extent that your other studies allow) to aid you in formulating your solutions, and you should not discuss the exam problems with anyone other than your instructor, except on trivial matters, until after the official exam time. Should you have any questions on the meaning or scope of the questions, or any other questions about the problems and their solutions, feel free to discuss them with your instructor via email or via Zoom (either in regular office hours or by request).

Each problem (number) has equal weight.

Exam 2 Problems:

1. Let X be a topological space. A set $B \subseteq X$ is called *nowhere dense* if the closure $\overline{B}$ of B has empty interior, i.e., $\text{int}(\overline{B}) = \emptyset$. Show that if $U \subseteq X$ is open, then $B = \overline{U} \setminus U$ is nowhere dense.

2. Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a continuous map. A *section* of f is a continuous map $\sigma : (Y, \mathcal{T}') \to (X, \mathcal{T})$ so that $f \circ \sigma = \text{Id}_Y$ (the identity map). Show that if f <u>has</u> a section, then f is a *quotient map*.

3. A space $(X, \mathcal{T})$ is called *locally path connected* if for every $x \in X$ and every $x \in U \in \mathcal{T}$ there is a $V \in \mathcal{T}$ so that $x \in V \subseteq U$ and V is path connected. Show that if $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are both locally path connected, then $X \times Y$, with the product topology, is locally path connected.

4. Suppose that X is a *normal* space, $C \subseteq X$ is a closed subset, and that $C \subseteq U_1 \cup U_2$, where U_1, U_2 are open in X. Show that there exist closed sets $C_1, C_2 \subseteq X$ with $C = C_1 \cup C_2$ and $C_1 \subseteq U_1$, $C_2 \subseteq U_2$. [Hint: The hypotheses allow you to build two disjoint closed sets...]

5. Show that if $f : X \to Y$ is a continuous map, and there are continuous maps $g, h : Y \to X$ with $f \circ g \simeq \text{Id}_Y$ and $h \circ f \simeq \text{Id}_X$, then f is a *homotopy equivalence*. That is, show that there is a <u>single</u> map $k : Y \to X$ with $f \circ k \simeq \text{Id}_Y$ and $k \circ f \simeq \text{Id}_X$. [Hint: note that our hypotheses actually give us <u>lots</u> of ways to write down maps from Y to X...]