Math 871 Problem Set 1

Starred (**) problems are due Thursday, August 27.

1. [Munkres, p.14, #2 (part)] For each statement below, determine whether or not it is true. If true, show why; if not, give an example demonstrating this.

   (a) For any sets A, B, C, $\{A \subseteq B$ and $A \subseteq C\}$ if and only if $A \subseteq B \cap C$.

(**) (b) For any sets A, B, C, $A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

   (c) For any sets A, B, C, D, $\{A \subseteq C$ and $B \subseteq D\}$ implies that $A \times C \subseteq B \times D$.

   (d) For any sets A, B, C, D, $A \times C \subseteq B \times D$ implies that $\{A \subseteq C$ and $B \subseteq D\}$.

   (e) For any sets A, B, C, D, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

(**) 2. If $A, C \subseteq X$ and $B, D \subseteq Y$, show how to express $(A \times B) \setminus (C \times D)$ as a union of Cartesian products (i.e., sets of the form $U \times V$).

3. (The flip side of the Axiom of Choice.) Show that if $A_i = \emptyset$ for some $i \in I$, then the Cartesian product $\prod_{i \in I} A_i = \emptyset$.

4. [Munkres, p.20, #1] Show that if $f : A \to B$ is a function, then

   (a) If $A_0 \subseteq A$, then $A_0 \subseteq f^{-1}(f(A_0))$; the sets are equal, if f is injective.

   (b) If $B_0 \subseteq B$, then $f(f^{-1}(B_0)) \subseteq B_0$; the sets are equal, if f is surjective.

5. [Munkres, p.20, #2 (part)] If $f : A \to B$ is a function, $A_0, A_1 \subseteq A$, and $B_0, B_1 \subseteq B$, then

   (a) $f^{-1}(B_0 \cap B_1) = f^{-1}(B_0) \cap f^{-1}(B_1)$

   (b) $f(A_0 \cap A_1) \subseteq f(A_0) \cap f(A_1)$, but equality does not always hold.

   (c) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$.

(**) (d) $f^{-1}(B_0 \setminus B_1) = f^{-1}(B_0) \setminus f^{-1}(B_1)$.

6. [Munkres, p.44, #7] If A and B are finite sets, show that the set $B^A = \{f : A \to B\}$ of all functions from A to B is also finite.

7. [Munlres, p.51, #5 (part)] For each of the following sets, determine whether or not it is countable:

(**) (a) $A = \{f : \{0, 1\} \to \mathbb{Z}$, all functions from $0, 1$ to $\mathbb{Z}$

(b) $B = \{f : \mathbb{N} \to \mathbb{N} \ : \ f(n) = 1$ for all $n \geq 4\}$

(c) $F = \{f : \mathbb{N} \to \{0, 1\} :$ there is $N \in \mathbb{N}$ with $f(n) = 0$ for all $n \geq N\}$, all eventually-0 functions.

(**) (d) $H = \{f : \mathbb{N} \to \mathbb{N} \ :$ there is $N \in \mathbb{N}$ with$f(n) = f(N)$ for all $n \geq N\}$, all eventually constant functions.

(e) $P = \{f : \mathbb{N} \to \mathbb{N} \ : \ n > m$ implies$f(n) > f(m)\}$, all increasing functions.

8. Let $P(A) = \{B \ : \ B \subseteq A\}$ (the *power set* of A), and suppose $f : A \to P(A)$ is a function. Show that the set $B = \{a \in A \ : \ a \notin f(a)\}$ is not in the image of f; conclude that f can never be surjective. What does this say about functions $g : P(A) \to A$?