Math 871 Problem Set 2

Starred (**) problems are due Thursday, September 2.

9. Show that if $d_1, d_2 : X \times X \to \mathbb{R}$ are both metrics on X, then the function $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = d_1(x,y) + d_2(x,y)$ is also a metric on X.

(**) 10. Show that if $d_1, d_2 : X \times X \to \mathbb{R}$ are both metrics on X, then the function $d : X \times X \to \mathbb{R}$ defined by $d(x,y) = \max\{d_1(x,y), d_2(x,y)\}$ is also a metric on X. What goes, or at least could go, wrong if "max" is replaced by "min"?

11. Show that if $f : (X,d) \to (Y,d')$ is a function between metric spaces, and there is a $\lambda \geq 0$ so that $d'(f(x), f(y)) \leq \lambda d(x,y)$ for every $x, y \in X$, then f is continuous.

[N.B.: Such functions are called Lipschitz (hence the "λ"...).]

(**) 12.(a) For a set X, and (fixed) $a \in X$, the excluded point topology on X is the collection of subsets $\mathcal{T}_a = \{U \subseteq X \ : \ a \notin U\} \cup \{X\}$. Show that $\mathcal{T}_a$ is a topology on X.

(**) (b) Show that if $(X, \mathcal{T}_a)$ and $(Y, \mathcal{T}_b)$ are excluded point topologies, then $f : (X, \mathcal{T}_a) \to (Y, \mathcal{T}_b)$ is continuous if and only if either f is constant or $f(a) = b$.

13. Show that the set $\mathcal{T} = \{(a, \infty) \ : \ a \in \mathbb{R}\} \cup \{[a, \infty) \ : \ a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ of subsets of $\mathbb{R}$ is a topology on $\mathbb{R}$. [Note that $\mathcal{T}$ can be thought of as the sets $U \subseteq \mathbb{R}$ so that whenever $a \in U$ and $b \geq a$, then $b \in U$.]

(**) 14. With the (excluded point) topology $\mathcal{T}_a$ on $\mathbb{R}$ from problem #12, and $\mathcal{T}'$ the "usual" topology on $\mathbb{R}$ (open sets are unions of neighborhoods), show that every continuous function $f : (\mathbb{R}, \mathcal{T}_a) \to (\mathbb{R}, \mathcal{T}')$ must be constant. [Hint: Suppose not! Show that f can't be continuous.]

15. With $\mathcal{T}$ the ('all rays'?) topology from problem #13 and $\mathcal{T}'$ the 'usual' topology on $\mathbb{R}$, show that every continuous function $f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}')$ is constant. [Hint: $\mathcal{T}$ doesn't contain very many disjoint sets...] Show, however, that there do exist continuous functions $f : (\mathbb{R}, \mathcal{T}') \to (\mathbb{R}, \mathcal{T})$ which are not constant. [Hint(?): How could $f^{-1}([a, \infty))$ and $f^{-1}((a, \infty))$ both be open?]