Math 871 Problem Set 3

Starred (**) problems are due Thursday, September 10.

16. If $d : X \times X \to \mathbb{R}$ is a metric on X, and $\mathcal{T}_d$ the resulting metric topology on X, show that for any $x_0 \in X$ the function

$$f : (X, \mathcal{T}_d) \to (\mathbb{R}, \mathcal{T}_{usual}) \text{ given by } f(x) = d(x_0, x)$$

is continuous.

17. [Munkres, p.83, #13.3] (a) Show that for any set X, the sets $\{U \subseteq X \ : \ X \setminus U \text{ is countable}\} \cup \{\emptyset\}$ form a topology on X. [This is (naturally) called the 'countable complement topology'.]

(b) Do the sets $\{U \subseteq X \ : \ X \setminus U \text{ is infinite}\} \cup \{\emptyset\}$ always form a topology on X ? Explain why or why not.

18. [Munkres, p.83, #13.4(a)] If $\mathcal{T}_\alpha$ are all topologies on the same set X, show that $\bigcap_\alpha \mathcal{T}_\alpha$ (the intersection of all of the topologies) is also a topology on X. Is $\bigcup_\alpha \mathcal{T}_\alpha$ (their union) a topology on X?

(**) 19. [A useful openness test.] If $(X, \mathcal{T})$ is a topological space, and $A \subseteq X$, show that $A \in \mathcal{T}$ if and only if

$$\text{for all } x \in A, \text{ there is a } U \in \mathcal{T} \text{ so that } x \in U \subseteq A$$

(**) 20. Show that if $\mathcal{T}_f$ is the finite-complement topology on the set X, then a non-constant continuous function $f : (X, \mathcal{T}_f) \to (X, \mathcal{T}_f)$ must be finite-to-one, i.e., $f^{-1}(\{a\})$ is finite for every $a \in \mathbb{R}$, and, conversely, that every finite-to-one function is continuous!

21. [Munkres, p.83, Problem #5] Show that if $\mathcal{B}$ is a basis for a topology on X, then $\mathcal{T}(\mathcal{B})$ is the intersection of all topologies that contain $\mathcal{B}$. Show that the analogous statement is true for the topology generated by a subbasis.

(**) 22. [Munkres, p.83, Problem #8] (a) Show that the collection $\mathcal{B} = \{(a,b) \ : \ a,b \in \mathbb{Q}\}$ is a basis, which generates the 'usual' topology on $\mathbb{R}$.

(**) (b) Show, by contrast, that the collection $\mathcal{B}' = \{[a,b) \ : \ a,b \in \mathbb{Q}\}$ is a basis for a topology on $\mathbb{R}$, but the topology it generates is strictly coarser than the 'lower limit' topology $\mathcal{T}_\ell$ on $\mathbb{R}$.

23. Show that if $\mathcal{B}$ and $\mathcal{B}'$ are both bases for topologies on X, then the set

$$\mathcal{B}'' = \{B \cap B' : B \in \mathcal{B} \text{ and } B' \in \mathcal{B}'\}$$

is also a basis for a topology on X , and that the topology it generates is the coarsest topology on X containing both $\mathcal{B}$ and $\mathcal{B}'$.