Math 871 Problem Set 4

Starred (**) problems are due Thursday, September 17.

24. Show that $\mathcal{B} = \{(a,\infty) \times (b,\infty) : a,b \in \mathbb{R}\}$ is a basis for a topology $\mathcal{T}$ on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, which is coarser than the usual Euclidean topology on $\mathbb{R}^2$. Show that $\mathcal{B}' = \{[a,\infty) \times [b,\infty) : a,b \in \mathbb{R}\}$ is a basis for a topology $\mathcal{T}'$ which is strictly finer than $\mathcal{T}$, and not comparable to the usual Euclidean topology.

(**) 25. [Munkres, p.92, Problem #5] If $\mathcal{T} \subseteq \mathcal{T}'$ are topologies on the set X and $\mathcal{O} \subseteq \mathcal{O}'$ are topologies on the set Y, show that the product topology $\mathcal{T} \times \mathcal{O}$ on $X \times Y$ is coarser than the topology $\mathcal{T}' \times \mathcal{O}'$. Is the converse result true [i.e., product topology coarser implies that the topologies on each factor are coarser]?

26. Show that if (X,d) and (Y,d') are metric spaces, then the product topology on $X \times Y$ is also a metric topology. [There are lots of (correct) choices of metric on $X \times Y$; you can take your cue from $\mathbb{R}^2$.]

(**) 27. Show that if (X,d) is a metric space, then the metric $d : X \times X \to \mathbb{R}$ is continuous (where $X \times X$ has the product topology, and $\mathbb{R}$ has the usual metric topology). Show, further, that the metric topology $\mathcal{T}$ is the <u>coarsest</u> topology on X for which d is continuous.

[Hint: show that if $\mathcal{T}' \subsetneq \mathcal{T}$, then $N_d(x_0,\epsilon) \notin \mathcal{T}'$ for some x_0 and $\epsilon > 0$; then a certain well-chosen composition $(X,\mathcal{T}') \to (X \times X, \mathcal{T}' \times \mathcal{T}') \xrightarrow{d} \mathbb{R}$, which <u>should</u> be continuous, won't be!]

(**) 28. [Munkres, p.112, Problem #10] Show that if A,B,X,Y are topological spaces, and $f : A \to X$ and $g : B \to Y$ are both continuous functions, then the function $h : A \times B \to X \times Y$, given by $h(a,b) = (f(a),g(b))$, is also continuous (using the product topologies on domain and codomain).

29. Theorem 15.1 of the text shows that if $(X,\mathcal{T})$ and $Y,\mathcal{T}'$ are topological spaces with bases $\mathcal{T} = \mathcal{T}(\mathcal{B})$ and $\mathcal{T}' = \mathcal{T}(\mathcal{B}')$, then $\mathcal{B} \times \mathcal{B}' = \{B \times B' : B \in \mathcal{B}, B' \in \mathcal{B}'\}$ is a basis for the product topology on $X \times Y$. [You should convince yourself of this!]

Show, by contrast, that, for an infinite Cartesian product, one natural generalization of this is (typically) false; if $\mathcal{T}_\alpha = \mathcal{T}(\mathcal{B}_\alpha)$ for all α, the sets

$$\prod_\alpha(\mathcal{B}_\alpha) = \{\prod_\alpha B_\alpha \ : \ B_\alpha \in \mathcal{B}_\alpha \text{ for all } \alpha\}$$

is a basis, but for the box topology on $\prod_\alpha X_\alpha$, not (typically) the product topology. What would the 'correct' basis be, to generate the product topology?

30. Show that if $\mathcal{T} \subseteq \mathcal{T}'$ are topologies on the set X, and $A \subseteq X$, then the subspace topology on A induced by $\mathcal{T}$ is coarser than the subspace topology induced by $\mathcal{T}'$. Find examples of topologies $\mathcal{T} \subsetneq \mathcal{T}'$ on $\mathbb{R}$ so that the topologies that they induce on $[0,1] \subseteq \mathbb{R}$ are the same.

31. Suppose that $(Y, \mathcal{T}')$ is a topological space, $f : X \to Y$ is a function, and we give X the coarsest topology $\mathcal{T}$ making $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ continuous. If $A \subseteq X$, show that the coarsest topology $\mathcal{T}''$ on A making $f|_A : (A, \mathcal{T}'') \to (Y, \mathcal{T}')$ continuous is the subspace topology that A inherits from $\mathcal{T}$.