**Math 871 Problem Set 6**

Starred (**) problems are due Thursday, October 1.

40. (a) Show that for every $c \in \mathbb{R}$ that the sets $\{(x,y) \in \mathbb{R}^2 \ : \ xy > c\}$ and $\{(x,y) \in \mathbb{R}^2 \ : \ xy < c\}$ are open in $\mathbb{R}^2$.

    (b) In a manner similar to problem #33, show that if $f, g : (X, \mathcal{T}) \to (\mathbb{R}, \text{usual})$ are both continuous, then the function $h : (X, \mathcal{T}) \to (\mathbb{R}, \text{usual})$ given by $h(x) = f(x) \cdot g(x)$ is also continuous.

41. [Munkres, p.111, #7(a)] Suppose that $f : \mathbb{R} \to \mathbb{R}$ is "continuous from the right", that is, for every $a \in \mathbb{R}$ we have $\lim_{x \to a^+} f(x) = f(a)$ (in the sense of calculus). Show that $f$ is continuous when thought of as a function from the *lower limit topology* $\mathcal{T}_\ell$ on $\mathbb{R}$ to the usual topology on $\mathbb{R}$.

(**) 42. [Munkres, p.112, #9(c)] A collection of subsets $\{A_\alpha\}_{\alpha in I}$ of $(X, \mathcal{T})$ is called *locally finite* of for every $x \in X$ there is a neighborhood $x \in U \in \mathcal{T}$ so that $U \cap A_\alpha$ is non-empty for only finitely many $\alpha$. Show that if $f : (X\mathcal{T}) \to (Y, \mathcal{T}')$ is a function, $X = \bigcup_{\alpha \in I} A_\alpha$, $\{A_\alpha\}_{\alpha \in I}$ is locally finite, all $A_\alpha$ are closed, <u>and</u> $f|_{A_\alpha} : (A_\alpha, \mathcal{T}_{A_\alpha}) \to (Y, \mathcal{T}')$ is continuous for all $\alpha \in I$, then $f$ is continuous.

    [Hint: Find a collection <u>open</u> sets whose union is $X$, each meeting only finitely many $A_\alpha$, and use <u>both</u> of our Pasting Lemmas! See the statements of parts (a) and (b) to help guide you...]

43. [Munkres, p.118, #8] For $i \in \mathbb{Z}_+$ let $a_i, b_i \in \mathbb{R}$ with $a_i > 0$ for all $i$, and let

    $f : \prod_i \mathbb{R} \to \prod_i \mathbb{R}$ be given by $f((x_i)_{i \in \mathbb{Z}_+}) = (a_i x_i + b_i)_{i \in \mathbb{Z}_+}$

    Show that $f$ is a homeomorphism, when $\prod_i \mathbb{R}$ is given the product topology (on both the domain and codomain). What happens when $\prod_i \mathbb{R}$ has the <u>box</u> topology?

44. Find an example of subspaces $A, B \subseteq \mathbb{R}$ (giving $\mathbb{R}$ the usual topology) for which there is a continuous bijection $f : A \to B$ whose inverse is **not** continuous.

(**) 45. Show that if $h : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is a homeomorphism, $A \subseteq X$, and $h(A) = B \subseteq Y$, then

$h|_A^B : (A, \mathcal{T}_A) \to (B, \mathcal{T}'_B)$ is also a homeomorphism.

46. [Munkres, p.101, #11] Show that if $(X_\alpha, \mathcal{T}_\alpha)$ are Hausdorff for all $\alpha$, then $\prod_\alpha X_\alpha$ is Hausdorff for <u>both</u> the product and box topologies.

(**) 47. Show that if $f, g : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ are both continuous, <u>and</u> $(Y, \mathcal{T}')$ is Hausdorff, then

$$C = \{x \in X : f(x) = g(x)\}$$

is a closed subset of $X$. [Show that the complement is open....]

48. [restatement of Munkres, p.112, #13] Show that if $f, g : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ are both continuous, $A \subseteq X$ is a <u>dense</u> subset of $X$, $(Y, \mathcal{T}')$ is Hausdorff, and $f|_A = g|_A$, then $f = g$.

[To paraphrase, a continuous function to a Hausdroff space is uniquely determined by its values on a dense subset.]