Math 871 Problem Set 7

Starred (**) problems are due Thursday, October 8.

49. Call a space $(X, \mathcal{T})$ *clumped* if for every $U, V \in \mathcal{T}$ with $U \neq \emptyset \neq V$ we have $U \cap V \neq \emptyset$. [I don't actually know if this property has a 'real' name...] Show that if $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous and $(X, \mathcal{T})$ is clumped, then the image $f(X) \subseteq Y$, with the subspace topology, is clumped. Conclude that 'clumped' is a homeomorphism invariant!

(**) 50. A space $(X, \mathcal{T})$ is called *path connected* if for every $x, y \in X$ there is a continuous map $\gamma : ([0, 1], \text{usual}) \to (X, \mathcal{T})$ with $\gamma(0) = x$ and $\gamma(1) = y$. Show that if $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous and $(X, \mathcal{T})$ is path connected, then the image $f(X) \subseteq Y$, with the subspace topology, is path connected. Conclude that 'path connected' is a homeomorphism invariant.

(**) 51. [Munkres, p.144, Problem #2]

(**) (a) Show that if $p : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous, and there is a continuous map $f : (Y, \mathcal{T}') \to (X, \mathcal{T})$ with $p \circ f = Id_Y$, then p is a quotient map.

(**) (b) For $A \subseteq X$, a *retraction* of X onto A is a continuous map $r : X \to A$ with $r|_A = Id_A$. Show that a retraction is a quotient map.

52. [Munkres, p.145, Problem #3] Let $X = \mathbb{R} \times \mathbb{R}$ with the product topology (and $\mathbb{R}$ given the usual topology), and $p : X \to (\mathbb{R}, \text{usual})$ the projection on the first coordinate, <u>and</u> let $A = \{(x, y) \in X \ : \ x \geq 0 \text{ or } y = 0\}$. Show that $p|_A : A \to \mathbb{R}$ is a quotient map, that is neither open nor closed.

53. [Munkres, p.145, Problem # 5] Suppose that $f : X \to Y$ is an open map, and $A \subseteq X$ is an open subset of X. Show that $f|_A : A \to Y$ is also an open map, so $f|_A : A \to f(A)$ is an open map (and hence a quotient map). Is the corresponding statement true if all instances of "open" are replaced with "closed"?

54 [Munkres, p.152, Problem #1] Show that if $\mathcal{T} \subseteq \mathcal{T}'$ are topologies on X and $(X, \mathcal{T}')$ is connected, then so is $(X, \mathcal{T})$.

55. Suppose that X is an infinite set. Which of the following topologies on X make X a connected space?

(a) the finite complement topology

(b) the excluded point topology (excluding $a \in X$).

(c) (for a fixed $a \in X$) the Fort topology

$$\mathcal{T} = \{U \subseteq X \ : \ a \notin U \text{ or } X \setminus U \text{ is finite}\}.$$

(d) the included point topology (including $b \in X$).

(e) the topology $\mathcal{T} = \{U \subseteq X \ : \ b \in U \text{ or } a \notin U\}$. [Note that there are two cases...]

(**) 56. Give an example of a space X and subset $A \subseteq X$ where $\text{int}(A)$ and $\text{cl}(A)$ are both connected, but A is not.