Math 871 Problem Set 8

Starred (**) problems are due Thursday, October 29.

61. [Munkres, p.152, Problem #11] Suppose that $p : X \to Y$ is a quotient map, Y is connected, and for every $y \in Y$ the set $p^{-1}(\{y\})$ is a connected subset of X. Show that X is connected.

    [Hint: start with a potential separation for X; show that their images would separate Y.]

62. Show that if $X_\alpha, \mathcal{T}_\alpha)$ are path-connected spaces, for $\alpha \in I$, then $\prod_{\alpha \in I} X_\alpha$, with the product topology, is also path-connected. Show, on the other hand, that there are examples where this is false, if we use the box topology instead.

(**) 63. [Munkres, p.152, Problem #9] Suppose that $A \subsetneq X$ and $B \subsetneq Y$ are *proper* subsets of X and Y. Show that if $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are both connected, then $(X \times Y) \setminus (A \times B)$ is a connected subset of $X \times Y$.

64. Show that if $\mathcal{T}$ and $\mathcal{T}'$ are toplogies on X, $(X, \mathcal{T}')$ is compact, and $\mathcal{T} \subseteq \mathcal{T}'$, then $(X, \mathcal{T})$ is compact.

(**) 65. Give an example of a space $(X, \mathcal{T})$ and subsets $A, B \subseteq X$ so that A and B are compact but $A \cap B$ is not.

    [Note: your example can't be Hausdorff!]

66. Let $(X, \mathcal{T})$ be a Hausdorff space and let

$$\mathcal{T}' = \{U \subseteq X \ : \ X \setminus U \subseteq X \text{ is compact}\} \cup \{\emptyset\}.$$

    Show that $\mathcal{T}'$ is a topology on X, and is coarser than $\mathcal{T}$. Show that, in general, the two topologies need not be equal. Do $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ have the same <u>compact</u> subsets?

67. [Munkres, p.171, Problem #5] Show that if $(X, \mathcal{T})$ is a Hausdorff space and $A, B \subseteq X$ are disjoint compact subsets of X, then there are subsets $U, V \in \mathcal{T}$ so that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$.

(**) 68. [Munkres, p.171, Problem #6] Show that if $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous and surjective, $(X, \mathcal{T})$ is compact, and $(Y, \mathcal{T}')$ is Hausdorff, then f is a closed map (and therefore a quotient map).