

## Math 872 Algebraic Topology

### Homework # 2

Starred (\*) problems due Thursday, February 1

6. Show that if there are maps  $f : X \rightarrow Y$  and  $g, h : Y \rightarrow X$  such that  $f \circ g : Y \rightarrow Y$  and  $h \circ f : X \rightarrow X$  are homotopic to  $I_Y$  and  $I_X$ , respectively, then  $f$  is a homotopy equivalence. Show that the same is true if  $f \circ g$  and  $h \circ f$  are homotopy equivalences.
- (\*) 7. Show that if  $X$  is a CW complex,  $A, B \subseteq X$  are subcomplexes,  $A \cup B = X$ , and  $A, B$  and  $A \cap B$  are contractible, then  $X$  is contractible.
8. Show that if  $f : \partial D^n \rightarrow X$  is the attaching map of an  $n$ -cell  $D^n$ , with  $n \geq 3$ , then the inclusion  $X \hookrightarrow X \cup_f D^n$  induces an isomorphism on  $\pi_1$ . Show that the same is true if we attach any (finite or infinite) collection of  $n \geq 3$  cells.
- (\*) 9. Let  $X =$  the space obtained from a cube  $J^3 = J \times J \times J$ ,  $J = [-1, 1]$ , by gluing opposite square faces to one another with a 90-degree righthand twist (e.g., glue  $J \times J \times \{-1\}$  to  $J \times J \times \{1\}$  by the map  $(x, y, -1) \mapsto (y, -x, 1)$ ). Describe a CW structure for  $X$  and compute a presentation for  $\pi_1(X)$ .
10. Starting with a 2-disk,  $D^2$ , with two small sub-disks deleted, show that there are essentially only two spaces obtained by identifying each of the two interior boundary circles to  $\partial D^2$  by homeomorphisms. Compute presentations for the fundamental groups of each, and (by abelianizing) show that the two groups are not isomorphic, so the two spaces are not homotopy equivalent.