Math 872 Algebraic Topology

Homework # 3

Starred (*) problems due Tuesday, February 13

11. Compute the fundamental group of the complement X of the coordinate axes in $\mathbb{R}^3$, i.e., $X = \{(x,y,z) \in \mathbb{R}^3 \ : \ |xy| + |xz| + |yz| \neq 0\}$. (Hint: Find a simpler space to compute the fundamental group of.)

12. Show that if $p_1 : \widetilde{X_1} \to X_1$ and $p_2 : \widetilde{X_2} \to X_2$ are covering maps, then the map $p = p_1 \times p_2 : \widetilde{X_1} \times \widetilde{X_2} \to X_1 \times X_2$, given by $p(x_1, x_2) = (p_1(x_1), p_2(x_2))$, is also a covering map.

(*) 13. Let $Y \subseteq \mathbb{R}^2$ denote the *quasi-circle*, given by $Y =$ $\{(x, \sin(1/x)) : x \in (0, \frac{1}{\pi}]\} \cup \{0\} \times [-1,1] \cup [-\frac{1}{\pi}, 0] \times \{0\} \cup \{(\frac{1}{\pi}\cos(t), -\frac{1}{\pi}\sin(t)) \colon 0 \leq t \leq \pi\}$ (See Hatcher, p.79, problem # 7 for an (approximate) picture.) The quotient map $q : Y \to Y/\sim$ that collapses the vertical subinterval $\{0\} \times [-1,1]$ to a point gives a space homeomorphic to S^1. Show that $\pi_1(Y) = \{1\}$ (hint: show that any path in Y is disjoint from $\{(x,y) \in Y \ : \ 0 < x < \epsilon\}$ for some $\epsilon > 0$), but the map q does not lift to the universal covering $p : \mathbb{R} \to S^1$. [This demonstrates that we cannot in general eliminate the hypothesis that Y be locally path-connected, in the lifting criterion.]

14. Let $p : \widetilde{X} \to X$ be a finite-sheeted covering space, with $\widetilde{X} \neq \emptyset$. Show that $\widetilde{X}$ is compact and Hausdorff $\Leftrightarrow$ X is compact and Hausdorff.

(*) 15. Let $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be covering spaces of path-connected, locally path connected spaces X and Y with $\widetilde{X}$ and $\widetilde{Y}$ simply-connected. Show that if X and Y are homotopy equivalent, then $\widetilde{X}$ and $\widetilde{Y}$ are homotopy equivalent. [Problem 6 from Problem Set # 1 might help.]