Math 872 Algebraic Topology

Problem Set # 4

Starred (*) problems due Tuesday, February 27

16. Show that $X = \mathbb{R}^2 \setminus \mathbb{Q}^2 \subseteq \mathbb{R}^2$ is path connected, and $\pi_1(X)$ is uncountable. (I.e., find uncountably many loops no two of which are homotopic to one another.)

17. Show that if $p : \widetilde{X} \to X$ is a covering map and $A \subseteq X$ is a subspace if X, then $p|_{p^{-1}(A)} : p^{-1}(A) \to A$ is also a covering map.

(*) 18. Find a pair of (finite) graphs (= 1-dim'l CW complexes with finitely many 0- and 1-cells) X_1 and X_2 that have a common finite-sheeted covering space $p_1 : X \to X_1$, $p_2 : X \to X_2$, but do *not* commonly cover another space, i.e., they are not both covering spaces of a single space Y.

19. Show that if a group G acts freely ($x = gx \Rightarrow g = 1$) and properly discontinuously (for all $x \in X$ there is a nbhd $\mathcal{U}$ of x such that $\{g \ : \ g(\mathcal{U}) \cap \mathcal{U} \neq \emptyset\}$ is finite) on a space X, then the quotient map $p : X \to X/G = X/\{x \sim gx \text{ for all } g \in G\}$ given by $p(x) = [x]$ is a covering map. In particular if X is Hausdorff and G is a finite group acting freely on X, then $p : X \to X/G$ is a covering map.
[Pointless remark: some people would write our quotient space as $G\backslash X$, since G is acting on the left, and so is being quotiented out from the left, although the Wikipedia entry on the matter, *http://en.wikipedia.org/wiki/Group_action*, agrees with us in this. Besides, as I just learned when TeXing this up, TeX doesn't like \ as a symbol, it asked me what the macro "\X" was supposed to mean ...?]

(*) 20. (Using covering spaces,) show that a finitely generated group G has only finitely many subgroups of a given index n. (Hint: do this first for a free group $F(m)$, then use the existence of a surjective homomorphism $\varphi : F(m) \to G$ for a suitable m.)

21. Show, using covering spaces, that the fundamental group of the closed orientable surface Σ of genus 2 is not abelian. (Hint: to show that for loops γ, η that $\gamma * \eta * \overline{\gamma} * \overline{\eta}$ isn't trivial, show that it (at least once) does not lift to a loop.)