Simplicial homology = singular homology: We have so far introduced two homologies; simplicial, H_*^Δ, whose computation "only" required some linear algebra, and singular, H_*, which is formally less difficult to work with, and which, you may suspect by now, is also becoming less difficult to compute... For Δ-complexes, these homology groups are the same, $H_n^\Delta(X) \cong H_n(X)$ for every X. In fact, the isomorphism is induced by the inclusion $C_n^\Delta(X) \subseteq C_n(X)$. And we have now assembled all of the tools necessary to prove this. Or almost; we need to note that most of the edifice we have built for singular homology could have been built for simplicial homology, including relative homology (for a sub-Δ-complex A of X), and a SES of chain groups, giving a LES sequence for the pair,

$$\cdots \to H_n^\Delta(A) \to H_n^\Delta(X) \to H_n^\Delta(X,A) \to H_{n-1}^\Delta(A) \to \cdots$$

The proof of the isomorphism between the two homologies proceeds by first showing that the inclusion induces an isomorphism on k-skeleta, $H_n^\Delta(X^{(k)}) \cong H_n(X^{(k)})$, and this goes by induction on k using the Five Lemma applied to the diagram

$$\begin{array}{ccccccccc} H_{n+1}^\Delta(X^{(k)},X^{(k-1)}) & \to & H_n^\Delta(X^{(k-1)}) & \to & H_n^\Delta(X^{(k)}) & \to & H_n^\Delta(X^{(k)},X^{(k-1)}) & \to & H_{n-1}^\Delta(X^{(k-1)}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{n+1}(X^{(k)},X^{(k-1)}) & \to & H_n(X^{(k-1)}) & \to & H_n(X^{(k)}) & \to & H_n(X^{(k)},X^{(k-1)}) & \to & H_{n-1}(X^{(k-1)}) \end{array}$$

The second and fifth vertical arrows are, by an inductive hypothesis, isomorphisms. The first and fourth vertical arrows are isomorphisms because, essentially, we can, in each case, identify these groups. $H_n(X^{(k)},X^{(k-1)}) \cong H_n(X^{(k)}/X^{(k-1)}) \cong \widetilde{H}_n(\vee S^k)$ are either 0 (for $n \neq k$) or $\oplus\mathbb{Z}$ (for $n = k$), one summand for each n-simplex in X. But the same is true for $H_n^\Delta(X^{(k)},X^{(k-1)})$; and for $n = k$ the generators are precisely the n-simplices of X. The inclusion-induced map takes generators to generators, so is an isomorphism. So by the Five Lemma, the middle rows are also isomorphisms, completing our inductive proof.

Returning to $H_n^\Delta(X) \xrightarrow{I_*} H_n(X)$, we wish now to show that this map is an isomorphism. Any $[z] \in H_n(X)$ is represented by a cycle $z = \sum a_i\sigma_i$ for $\sigma_i : \Delta^n \to X$. But each $\sigma_i(\Delta^n)$ is a compact subset of X, and so meets only finitely-many cells of X. This is true for every singular simplex, and so there is a k for which all of the simplices map into $X^{(k)}$, and so we may treat $z \in C_n(X^{(k)}$. Thought of in this way, it is still a cycle, and so $[z] \in H_n(X^{(k)}) \cong H_n^\Delta(X^{(k)})$ so there is a $z' in C_n^\Delta(X^{(k)})$ and a $w \in C_{n+1}(X^{(k)})$ with $i_\# z' - z = \partial w$. But thinking of $z' in C_n^\Delta(X)$ and $w \in C_{n+1}(X)$, we have the same equality, so $[z'] \in H_n^\Delta(X)$ and $i_*[z'] = [z]$. So i_* is surjective. If $i_*([z]) = 0$, then the cycle $z = \sum a_i\sigma_i$ is a sum of characteristic maps of n-simplices of X, and so can be thought of as an element of $C_n^\Delta(X^{n)})$. Being 0 in $H_n(X)$, $z = \partial w$ for some $w \in C_{n+1}(X)$. But as before, $w \in C_n(X^{r)})$ for some r, and so thought of as an element of the image of the isomorphism $i_* : H_n^\Delta(X^{(r)}) \to H_n(X^{(r)})$, $i_*([z]) = 0$, so $[z] = 0$. So $z = \partial u$ for some $u \in C_{n+1}^\Delta(X^{r)}) \subseteq C_{n+1}^\Delta(X)$. So $[z] = 0$ in $H_n^\Delta(X)$. Consequently, simplicial and singular homology groups are isomorphic.

The isomorphism between simplicial and singular homology provides very quick proofs of several results about singular homology, which would other would require some effort:

If the Δ-complex X has no simplices in dimension greater than n, then $H_i(X) = 0$ for all $i > n$.

This is because the simplicial chain groups $C_i^\Delta(X)$ are 0, so $H_i^\Delta(X) = 0$.

If for each n, the Δ-complex X has finitely many n-simplices, then $H_n(X)$ is finitely generated for every n.

This is because the simplicial chain groups $C_n^\Delta(X)$ are all finitely generated, so $H_n^\Delta(X)$, being a quotient of a subgroup, is also finitely generated. [We are using here that the number of generators of a subgroup H of an *abelian* group G is no larger than that for G; this is not true for groups in general!]

Some more topological results with homological proofs: The Klein bottle and real projective plane cannot embed in $\mathbb{R}^3$. This is because a surface Σ embedded in $\mathbb{R}^3$ has a (the proper word is *normal*) neighborhood $N(\Sigma)$, which deformation retracts to Σ; literally, it is all points within a (uniformly) short distance in the normal direction from the point on the surface Σ. Our non-embeddedness result follows (by contradiction) from applying Mayer-Vietoris to the pair $(A,B) = (\overline{N(\Sigma)}, \overline{\mathbb{R}^3 \setminus N(\Sigma)})$, whose intersection is the boundary $F = \partial N(\Sigma)$ of the normal neighborhood. The point, though, is that F is an orientable surface; the outward normal (pointing away from $N(\Sigma)$) at every point, taken as the first vector of a right-handed orientation of $\mathbb{R}^3$ allows us to use the other two vectors as an orientation of the surface. So F is one of the surface F_g above whose homologies we just computed. This gives the LES $\widetilde{H}_2(\mathbb{R}^3) \to \widetilde{H}_1(F) \to \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \to \widetilde{H}_1(\mathbb{R}^3)$ which renders as $0 \to \mathbb{Z}^{2g} \to \widetilde{H}(\Sigma) \oplus G \to 0$, i.e., $\mathbb{Z}^{2g} \cong \widetilde{H}(\Sigma) \oplus G$. But for the Klein bottle and projective plane (or any closed, non-orientable surface for that matter), $\widetilde{H}_1(\Sigma)$ has torsion, so it cannot be the direct summand of a torsion-free group! So no such embedding exists. This result holds more generally for any 2-complex K whose (it turns out it would have to be first) homology has torsion; any embedding into $\mathbb{R}^3$ would have a neighborhood deformation retracting to K, with boundary a (for the exact same reasons as above) closed orientable surface.

Invariance of Domain: If $\mathcal{U} \subseteq \mathbb{R}^n$ and $f : \mathcal{U} \to \mathbb{R}^n$ is continuous and injective, then $f(\mathcal{U}) \subseteq \mathbb{R}^n$ is open.

We will approach this through the **Brouwer-Jordan Separation Theorem:** an embedded $(n-1)$-sphere in $\mathbb{R}^n$ separates $\mathbb{R}^n$ into two path components. And for this we need to do a slightly unusual homology calculation:

For $k < n$ and $h : I^k \to S^n$ an embedding of a k-cube in to the n-sphere, $\widetilde{H}_i(S^n \setminus h(I^k)) = 0$ for all i.

Here $I = [-1,1]$. The proof proceeds by induction on k. For $k = 0$, $S^n \setminus h(I^k) \cong \mathbb{R}^n$, and the result follows. Now suppose the result os true for all embeddings of $C = I^{k-1}$, but is false for some embedding $h : I^k \to S^n$ and some i. Then if we divide the cube along its last coordinate, say, as $I^{k-1} \times [-1,0] = C \times [-1,0]$ and $C \times [0,1]$, we can set $A = S^n \setminus h(C \times [-1,0])$, $B = S^n \setminus h(C \times [0,1])$, $A \cup B = S^n \setminus h(C \times \{0\})$, and $A \cap B = S^n \setminus h(I^k)$. These sets are all open, since the image under h of the various sets is compact, hence closed. By hypothesis, $A \cup B = S^n \setminus h(C \times \{0\})$ has trivial reduced homology, while $A \cap B = S^n \setminus h(I^k)$ has non-trivial reduced homology in some dimension i. Then the Mayer-Vietoris sequence

$$\cdots \to \widetilde{H}_{i+1}(A \cup B) \to \widetilde{H}_i(A \cap B) \to \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \to \widetilde{H}_i(A \cup B) \to \cdots$$

reads $0 \to \widetilde{H}_i(A \cap B) \to \widetilde{H}_i(A) \oplus \widetilde{H}_i(B) \to 0$ so $\widetilde{H}_i(A \cap B) \cong \widetilde{H}_i(A) \oplus \widetilde{H}_i(B)$, so at least one of the groups on the right must be non-trivial, as well. WOLOG $\widetilde{H}_i(B) = \widetilde{H}(S^n \setminus h(C \times [0,1])) \neq 0$. Even more, choosing (once and for all) a non-zero element $[z] \in \widetilde{H}_I(A \cap B)$, snce its image in the direct sum is non-zero, it's coordinate in (say) $\widetilde{H}_i(B)$ is non-zero.

Continuing with: For $k < n$ and $h : I^k \to S^n$ an embedding of a k-cube in to the n-sphere, $\widetilde{H}_i(S^n \setminus h(I^k)) = 0$ for all i.

We've shown how we can throw away half of the cube without losing a (chosen) non-zero homology element. Now we continue inductively, cutting $C \times [0,1]$ in two along the last coordinate as $C \times [0,1/2], C \times [1/2,1]$ and repeat the same argument. We fnd that $\widetilde{H}_i(S^n \setminus h(C \times [a,b])) \neq 0$, and $[z]$ maps to a non-zero element under the inclusion-induced homomorphism.. Continuing inductively, we find a sequence of nested intervals $I_n = [a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$ whose lengths tend to zero (so $a_n, b_n \to x_0 \in I$ as $n \to \infty$), and injective inclusion-induced maps

$$0 \neq \widetilde{H}_i(S^n \setminus h(I^n) \to \cdots \to \widetilde{H}_i(S^n \setminus h(C \times I_n) \to \widetilde{H}_i(S^n \setminus h(C \times I_{n+1})$$

all of which send a certain non-zero element $[z] \in \widetilde{H}_i(S^n \setminus h(I^n)$ to a non-zero element, and all of which have an inclusion-induced map to $\widetilde{H}_i(S^n \setminus h(C \times \{x_0\}) = 0$. So there is a non-trivial element $[z] \in \widetilde{H}_i(S^n \setminus h(I^n)$ which remains non-zero in all $\widetilde{H}_i(S^n \setminus h(C \times I_n))$, but is zero in $\widetilde{H}_i(S^n \setminus h(C \times \{x_0\})$. Consequently, $z\partial w$ for some chain $w = \sum a_j \sigma_j^{i+1} \in C_{i+1}(S^n \setminus h(C \times \{x_0\}))$. Each singular simplex, however, is a map $\sigma_j^{i+1} : \Delta^{i+1} \to S^n \setminus h(C \times \{x_0\})$, and so has compact image. But the sets $S^n \setminus h(C \times I_n)$ form a nested open cover of $S^n \setminus h(C \times \{x_0\})$, and so of $\sigma_j^{i+1}(\Delta^{i+1})$, and so there is an n_j with $\sigma_j^{i+1}(\Delta^{i+1}) \subseteq S^n \setminus h(C \times I_{n_j})$. Then setting $N = \max\{n_j\}$, we have $\sigma_j^{i+1} : \Delta^{i+1} \to S^n \setminus h(C \times I_N)$ for every j, so $w \in C_{i+1}(S^n \setminus h(C \times I_N)$, so $0 = [z] \in \widetilde{H}_i(S^n \setminus h(C \times I_N)$, a contradiction. So $\widetilde{H}_i(S^n \setminus h(I^k)) = 0$, and our inductive step is proved.

One immediate consequence of this is that if $h : S^k \to S^n$ is an embedding of the k-sphere into the n-sphere, then thinking of S^k as the union of its upper and lower hemispheres, D^k_+, D^k_-, each of which is homeomorphic to I^k, we have $D^k_+ \cap D^k_- = S^{k-1}$, the equatorial $(k-1)$-sphere, and so by Mayer-Vietoris we have

$$\cdots \to \widetilde{H}_{i+1}(S^n \setminus h(D^k_-)) \oplus \widetilde{H}_{i+1}(S^n \setminus h(D^k_+) \to \widetilde{H}_{i+1}(S^n \setminus h(S^{k-1})) \to \widetilde{H}_i(S^n \setminus h(S^k)) \to \widetilde{H}_i(S^n \setminus h(D^k_-)) \oplus \widetilde{H}_i(S^n \setminus h(D^k_+) \to \cdots$$

i.e., $\widetilde{H}_i(S^n \setminus h(S^k)) \cong \widetilde{H}_{i+1}(S^n \setminus h(S^{k-1})) \cong \cdots \cong \widetilde{H}_{i+k}(S^n \setminus h(S^0)) \cong \widetilde{H}_{i+k}(S^{n-1})$, since $S^0 = 2$ points, and so $S^n \setminus h(S^0) \cong S^{n-1} \times \mathbb{R} \sim S^{n-1}$. So $\widetilde{H}_i(S^n \setminus h(S^k)) = 0$ unless $i + k = n - 1$ (i.e., $i = n - k - 1$), when it is $\mathbb{Z}$.

In particular, $\widetilde{H}_0(S^n \setminus h(S^{n-1})) = \mathbb{Z}$, so we have the Jordan-Brouwer Separation Theorem: every embedded S^{n-1} in S^n has two complementary path-components A, B . With a little work, one can show that $\overline{A} \cap \overline{B} = h(S^{n-1}$, so the $(n-1)$-sphere is the frontier of each complementary component. [Removing a point from S^n to get $\mathbb{R}^n$ does not change the conclusion (for $n > 1$); a point does not disconnect an open subset of S^n.]

When $n = 2$, the Jordan Curve Theorem (as it is then called) has the additional consequence that the closure of each complementary region is a compact 2-disk, each having the embedded circle $h(S^1)$ as its boundary. This stronger result does not extend to higher dimensions, without putting extra restrictions on the embedding. This was shown by Alexander (shortly after publishing an incorrect proof without restrictions) for $n = 3$; these examples are known as the Alexander horned spheres.

To prove Invariance of Domain, let $\mathcal{U} \subseteq \mathbb{R}^n \subseteq S^n$ be an open set, and $f : \mathcal{U} \to \mathbb{R}^n \hookrightarrow S^n$ be injective and continuous. It suffices to show, for every $x \in \mathcal{U}$, that there is an open neighborhood $\mathcal{V}$ with $f(x) \subseteq \mathcal{V} \subseteq f(\mathcal{U})$. Since $\mathcal{U}$ is open, there is an open ball B^n centered at x whose closure D^n is contained in $\mathcal{U}$. f is then an embedding of $\partial D^n = S^{n-1}$ into S^n, and of $D^n \cong I^n$ into S^n. By our calculations above, $S^n \setminus f(S^{n-1})$ has two path components A, B; being an open set and contained in a locally path-connected space, these are also the connected components of the complement. But our calculations above also show that $S^n \setminus f(D^n)$ is path-connected, hence connected, and $f(B^n)$, being the image of a connected set, is connected. Since $f(B^n) \cup (S^n \setminus f(D^n)) = S^n \setminus f(S^{n-1} = A \cup B)$, it follows that $f(B^n) = A$ and $S^n \setminus f(D^n) = B$ (or vice versa). In particular, $f(B^n)$ is open, forming an open subset of $f(\mathcal{U})$ containing $f(x)$, as desired.

Invariance of Domain in turn implies the "other" invariance of domain; if $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and injective, then $n \leq m$, since if not, then composition of f with the inclusion $i : \mathbb{R}^m \to \mathbb{R}^n$, $i(x_1, \ldots, x_m) = (x_1, \ldots, x_m, 0, \ldots, 0)$ is injective and continuous with non-open image (it lies in a hyperplane in $\mathbb{R}^n$), a contradiction.

This also gives the more elementary: if $\mathbb{R}^n \cong \mathbb{R}^m$, via h, then $n = m$. Another proof: by composing with a translation, that $h(0) = 0$, and then we have $(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \mathbb{R}^m, (\mathbb{R}^m \setminus 0)$, which gives

$$\begin{aligned} \widetilde{H}_i(S^{n-1}) &\cong H_{i+1}(\mathbb{D}^n, \partial\mathbb{D}^n) \cong H_{i+1}(\mathbb{D}^n, \mathbb{D}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong H_{i+1}(\mathbb{R}^m, \mathbb{R}^m \setminus 0) \\ &\cong H_{i+1}(\mathbb{D}^m, \mathbb{D}^m \setminus 0) \cong H_{i+1}(\mathbb{D}^m, \partial\mathbb{D}^m) \cong \widetilde{H}_i(S^{m-1}) \end{aligned}$$

Setting $i = n - 1$ gives the result, since $\widetilde{H}_{n-1}(S^{m-1}) \cong \mathbb{Z}$ implies $n - 1 = m - 1$.

Homology and homotopy groups: There are connections between homology groups and the fundamental (and higher) homotopy groups, provided by what is known as the *Hurewicz map* $H : \pi_n(X, x_0) \to H_n(X)$. or $n = 1$ (higher n are similar) the idea is that elements of $\pi_1(X)$ are loops, which can be thought of as maps $\gamma : S^1 \to X$ (or more precisely, mapping into the path component containing x_0), inducing a map $\gamma_* : \mathbb{Z} = H_1(S^1) \to H_1(X)$. We define $H([\gamma]) = \gamma_*(1)$. Because homotopic maps give the same induced map on homology, this really is well-defined map on homotopy classes, i.e. from $\pi_1(X)$ to $H_1(X)$. [A different view: a loop $\gamma : (I, \partial I) \to (X, x_0)$ defines a singular 1-chain which, being a loop, has zero boundary, so is a 1-cycle. Since based homotopic maps give homologous chains (essentially by the same homotopy invariance property above), we get a well-defined map $\pi_1(X, x_0) \to H_1(X)$.

Since as 1-chains, the concatenation $\gamma * \delta$ of two loops is homologous to the sum $\gamma + \delta$ - the map $K : I \times I \to X$ given by $K(s, t) = (\gamma * \delta)(s)$, after crushing the left and right vertical boundaries to points, can be thought of as a singular 2-simplex with boundary $\gamma + \delta - (\gamma * \delta)$ - the map H is a homomorphism.

When X is path-connected, this map $H : \pi_1(X) \to H_1(X)$ is onto . [When it isn't it maps onto the summand of $H_1(X)$ corresponding to the path component containing our chosen basepoint.] To see this, note that any cycle $z \in Z_1(X)$ can be represented as a sum of singular 1-simplices $\sum \sigma_i^1$, i.e. we can (by reversing the orientations on simplices to make coefficient positive, and then writing a multiple of a simplex as a sum of simplices) assume all coefficients in our sum are 1. Then $0 = \partial z = \sum(\sigma_i^1(0,1) - \sigma_i^1(1,0))$ means that, starting with any positive term, we can match it with a negative term to cancel that term, which is paired with a postive term, having a matching negative term, etc., until the initial positive term is cancelled. This sub-chain represents a collection of paths which concatenate to a loop, so $z =$ (this loop) + (the remainiung terms) . Induction implies that z can be written as a sum of (sums of paths forming loops), which is (as above) homologous to the sum of loops. Choosing paths from the start of these loops to our chosen basepoint (which is the only place where we use path connectedness, we can concatenate the based loops $\overline{\gamma} * \sigma * \gamma$ to a single based loop η, which under H is sent to a chain homologous to z. So $H[\eta] = [z]$.

Since $H_1(X)$ is abelian (and $\pi_1(X)$ need not be), the kernel of H contains the commutator subgroup $[\pi_1(X), \pi_1(X)]$. We now show that, if X is path connected, H induces an isomorphism $H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)]$. To show this, it remains to show that $\ker(H) \subseteq [\pi_1(X), \pi_1(X)]$. Or put differently, the ineduced map from $\pi_1(X)_{ab} = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ (i.e., $\pi_1(X)$, written using additive notation) to $H_1(X)$ is injective. So suppose $[\gamma] \in \pi_1(X)$ and, thought of as a singular 1-simplex, $\gamma = \partial w$ for some 2-simplex $w = \sum a_i \sigma_i^2$. As before, we may assume that all $a_i = 1$, by reversing orientation and writing multiples as sums. By adding "tails" from each image of a vertex of each σ_i^2 to our chosen basepoint x_0, we may assume that the image of every face of Δ^2, under the σ_i , is a loop at x_0 (by essentially replacing each σ_i with a τ_i which first collapses little triangle at each vertex to arcs, maps the resulting central triangle via σ_i, and the arcs via the paths).

Once we have made this slight alteration, the equation $\gamma = \partial w = \displaystyle\sum_{i=1}^{n} \sum_{j=0}^{2} \partial_j \sigma_i = 0$ makes sense (and is true) in both ($C_1(X)$ hence $Z_1(X)$ hence) $H_1(X)$ and $\pi_1(X)_{ab}$, the first essentially by definition and the second because all of the $\partial_j \sigma_i$ are loops at x_0 and, in $\pi_1(X)$, $(\partial_0 \sigma_i)\overline{\partial_1 \sigma_i}(\partial_2 \sigma_i)$ is null-homotopic, so is trivial in $\pi_1(X)$. Written additively, this means that in $\pi_1(X)_{ab}$, $\partial_0 \sigma_i - \partial_1 \sigma_i + \partial_2 \sigma_i = 0$. So $\gamma = 0$ in $\pi_1(X)_{ab}$, as desired.

he Hurewicz map $H : \pi_1(X) \to H_1(X)$ induces, when X is path-connected, an isomorphism from $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ to $H_1(X)$. This result can be used in two ways; knowing a (presentation for) $\pi_1(X)$ allows us to compute $H_1(X)$, by writing the relators additively, giving $H_1(X)$ as the free abelian group on the generators, modulo the kernel of the "presentation matrix" given by the resulting linear equations. Conversely, knowing $H_1(X)$ provides information about $\pi_1(X)$. For example, a calculation on the way to invariance of domain implied that for every knot K in S^3 (i.e., the image of an embedding $h : S^1 \hookrightarrow S^3$), $H_1(S^3 \setminus K) \cong bbz$. This implies that the abelianization of $G_K = \pi_1(S^3 \setminus K)$ (i.e., the largest abelian quotient of G_K is $\mathbb{Z}$. But this in turn implies that for every integer $n \geq 2$, there is a <u>unique</u> surjective homomorphism $G_K \to \mathbb{Z}_n$, since such a homomorphism must factor through the abelianization, and there is exactly one surjective homomorphism $\mathbb{Z} \to \mathbb{Z}_n$! Consequently, there is a unique (normal) subroup (the kernel of this homomorphism) $K_n \subseteq G_K$ with quotient $\mathbb{Z}_n$. Using the Galois correspondence, there is a (unique) covering space X_n of $X = S^3 \setminus K$ corresponding to K_n, called the n-fold cyclic covering of K . This space is determined by K and n, and so its homology groups are determined by the same data. And even though homology cannot distinguish between two knot complements, K, K', it might be the case that homology can distinguish between their cyclic coverings. Consequently, if $H_1(X_n) \not\cong H_1(X'_n)$, then K and K' have non-homeomorphic complement, and so represent "different" embeddings, hence different knots. In practice, one can compute presentations for $\pi_1(X_n)$ (in several different ways), and so one can compute $H_1(X_n)$, providing an effective way to use homology to distinguish knots! This approach was ultimately formalized (by Alexander) into a polynomial invariant of knots, known as the Alexander polynomial.

Computing the homology of the cyclic coverings can be done in several ways. The Reidemeister-Schreier method will allow one to compute a presentation for the kernel of a homomorphism $\varphi : G \to H$, given a presentation of G and a *transversal* of the map, which is a representative of each coset of G modulo the kernel. Abelianizing this will give homology computation. Another approach uses *Seifert surfaces*, orientable surfaces with $\partial\Sigma = K$, to cut $S^3 \setminus K$ open along. Writing $S^3 \setminus K = (S^3 \setminus N(\Sigma)) \cup N(\Sigma)$ allows us to use Mayer-Vietoris to compute homology. But the cyclic covering spaces can be built by "unwinding" this view of $S^3 \setminus K$; instead of gluing the two ends of $N(K)$ to the same $S^3 \setminus N(\Sigma)$, we can take n copies of $S^3 \setminus N(\Sigma)$ and glue them together in a circle. Mayer-Vietoris again tells us how to compute the homology of the resulting space. Details may be found on the accompanying pages taken from Rolfsen's "Knots and Links".