The Fundamanetal Group:

Basic concept: homotopy

$f, g : X \to Y$ are *homotopic* if there is $H : X \times I \to Y$ with $H(x,0) = f(x)$, $H(x,1) = g(x)$ for all $x \in X$. **[equivalence relation]**

Loops at x_0: $\gamma : (I, \partial I) \to (X, \{x_0\})$ **[map of pairs]** $\quad I = [0,1]$

Elements of $\pi_1(X, x_0)$ are homotopy classes of loops at x_0 **[homotopy of pairs]**

multiplication = concatenation: $\alpha * \beta(t) = \begin{cases} \alpha(2t), & \text{if } t \leq 1/2 \\ \beta(2t-1) & \text{if } t \geq 1/2 \end{cases}$

Multiplication of homotopy classes is well-defined: $\alpha \simeq \alpha'$,$\beta \simeq \beta'$
$\Rightarrow \alpha * \beta \simeq \alpha' * \beta'$

The (class of the) constant map is the identity: $c_{x_0} * \gamma \simeq \gamma \simeq \gamma * c_{x_0}$

Associativity from $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$

Inverse is (the class of) the reverse of γ : $\overline{\gamma}(t) = \gamma(1-t)$

$$\gamma * \overline{\gamma} \simeq c_{x_0} \simeq \overline{\gamma} * \gamma$$

Induced homomorphisms: A map $f : (X, x_0) \to (Y, y_0)$ induces a homomorphism $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $f_*[\gamma] = [f \circ \gamma]$.

(In)dependence on basepoint x_0: if γ is a path from x_0 to x_1, then $\widehat{\gamma}([\alpha]) = [\gamma * \alpha * \overline{\gamma}]$ is an isomorphism $\widehat{\gamma} : \pi_1(X, x_0) \to \pi_1(X, x_1)$. (Inverse is $\widehat{(\overline{\gamma})} = (\widehat{\gamma})^{-1}$)