The Lebesgue Number Theorem:

If (X, d) is a compact metric space (in our applications, it is always a compact subset of Euclidean space), and $\{\mathcal{U}_i\}$ is an open cover of X, then there is an $\epsilon > 0$ so that for every $x \in X$, its ϵ-neighborhood $N_d(x, \epsilon)$ is contained in $\mathcal{U}_i$ for some i.

Proof: If not, then for every $n \in \mathbb{N}$ there is an $x_n \in X$ whose $1/n$-neighborhood is contained in no $\mathcal{U}_i$; that is, for every $i \in I$, there is an $x_{n,i}$ with $d(x_n, x_{n,i}) < 1/n$ and $x_{n,i} \notin \mathcal{U}_i$, so $x_{n,i} \in C_i = X \setminus \mathcal{U}_i$, a closed set. But since X is compact, there is a convergent subsequence of the x_n; $x_{n_k} \to y \in X$.

[Proof: if not, then no point is the limit of a subsequence, so for every $x \in X$ there is an $\epsilon(x) > 0$ and an $N = N(x)$ so that $n \geq N$ implies $x_n \notin N_d(x, \epsilon(x))$. But these neighborhoods cover X, so a finite number of them do; for any n greater than the maximum of the associated $N(x)$'s x_n lies in none of the neighborhoods, a contradiction, since $x_n \in X =$ the union of these neighborhoods.]

But then since $d(x_{n_k}, y) \to 0$ and $d(x_{n_k}, x_{n_k,i}) \to 0$, for every i the $x_{n_k,i}$ also converge to y; since the $x_{n_k,i}$ all lie in the closed set C_i, so does y. So $y \in C_i$ for all i, so $y \notin \mathcal{U}_i$ for all i, a contradiction, since the $\mathcal{U}_i$ cover X. So some $\epsilon > 0$, a *Lebesgue number* for the covering, must exist.