Applications: the Brouwer Fixed Point Theorem

Calculus: $f : I \to I$ has a fixed point; $f(x_0) = x_0$ [Pf: apply IVT to $g(x) = f(x) - x$.]

Brouwer: Every map $f : D^2 \to D^2$ has a fixed point.

Proof: If not, then $f(x) \neq x$ for all $x \in D^2$. Construct a retraction $r : D^2 \to \partial D^2$ by sending $x \in D^2$ to the point on ∂D^2 lying on the ray from $f(x)$ to x. Formally:

Send x to $y = x + t(f(x) - x)$, $t \leq 0$ such that $||x + t(f(x) - x)|| = 1$.
$||x + t(f(x) - x)||^2 - 1 = \langle x + t(f(x) - x), x + t(f(x) - x)\rangle - 1 =$
$||f(x) - x||^2 t^2 + 2\langle x, f(x) - x\rangle t + (||x||^2 - 1) = at^2 + bt + c = 0$ and $t \leq 0$.
I.e. (note that $c \leq 0$ and $a > 0$), $t = (-b - \sqrt{b^2 - 4ac})/2a =$
$(-\langle x, f(x) - x\rangle - \sqrt{\langle x, f(x) - x\rangle^2 - ||f(x) - x||^2(||x||^2 - 1)})/||f(x) - x||^2$, so

$$r(x) = x + \frac{-\langle x, f(x) - x\rangle - \sqrt{\langle x, f(x) - x\rangle^2 - ||f(x) - x||^2(||x||^2 - 1)}}{||f(x) - x||^2}(f(x) - x)$$

which, since $||x - f(x)||$ is bounded away from 0 (it has a positive minimum on D^2), is continuous. Check: if $||x|| = 1$, then $r(x) = x$ (because $\langle x, f(x) - x\rangle < 0$). But a retraction induces a surjective homomorphism on π_1, so r_* is a surjection from $\pi_1(D^2) = 1$ to $\pi_1(\partial D^2) = \pi_1(S^1) = \mathbb{Z}$, a contradiction. So f must have a fixed point.

Basic idea: if no fixed point, then a new map that we build has a property (retraction) which translates into the algebra (surjection) to something which we know can't be true.

The exact same proof will apply in higher dimensions (every $f : D^n \to D^n$ has a fixed point), once we build an algebraic gadget H ("the $(n-1)^{\text{st}}$ homology group") for which $H(D^n) = \{1\}$ and $H(\partial D^n) = H(S^{n-1}) \neq \{1\}$.