

Another non-trivial presentation of the trivial group

$$\begin{aligned} G &= \langle a, b, c \mid bbab^{-1}a^{-1}, ccbc^{-1}b^{-1}, aaca^{-1}c^{-1} \rangle \\ &= \langle a, b, c \mid aba^{-1} = b^2, bcb^{-1} = c^2, cac^{-1} = a^2 \rangle \end{aligned}$$

Then: $aba^{-1} = b^2$, so $(aba^{-1})^{-1} = ab^{-1}a^{-1} = b^{-2}$,
so $ab^{-1} = b^{-2}a$. Also,
 $b^2a = ab$, so $bab^{-1} = b^{-1}(b^2a)b = b^{-1}(ab)b^{-1} = b^{-1}a$.

Also,

$$b^2ab^{-2} = b(bab^{-1})b^{-1} = bb^{-1}ab^{-1} = ab^{-1} = b^{-2}a.$$

Similarly, $c^2b = bc$, so $cb = c^{-1}bc$, and $c^2bc^{-2} = c^{-2}b$.

Then: $cac^{-1} = a^2$, so $b(cac^{-1})b^{-1} = b(a^2)b^{-1}$. But

$$\begin{aligned} b(cac^{-1})b^{-1} &= (bc)a(bc)^{-1} = (c^2b)a(c^2b)^{-1} = c^2(bab^{-1})c^{-2} \\ &= c^2(b^{-1}a)c^{-2} = (c^2b^{-1}c^{-2})(c^2ac^{-2}) \\ &= (c^2bc^{-2})^{-1}c(cac^{-1})c^{-1} = (c^{-2}b)^{-1}c(a^2)c^{-1} \\ &= b^{-1}c^2(cac^{-1})^2 = b^{-1}c^2(a^2)^2 = b^{-1}c^2a^4 \end{aligned}$$

and

$$\begin{aligned} b(a^2)b^{-1} &= (bab^{-1})^2 = (b^{-1}a)^2 = b^{-1}ab^{-1}a = b^{-1}(ab^{-1})a \\ &= b^{-1}(b^{-2}a)a = b^{-3}a^2, \end{aligned}$$

$$\text{so } b^{-1}c^2a^4 = b^{-3}a^2,$$

$$\text{so } c^2 = b(b^{-1}c^2a^4)a^{-4} = b(b^{-3}a^2)a^{-4} = b^{-2}a^{-2}.$$

$$\text{So } c^{-2} = (c^2)^{-1} = (b^{-2}a^{-2})^{-1} = a^2b^2.$$

$$\text{Then } bc = c^{-1}(cb)c = c^{-1}(c^{-1}bc)c = c^{-2}bc^2$$

$$= (a^2b^2)b(b^{-2}a^{-2}) = a^2ba^{-2} = a(aba^{-1})a^{-1}$$

$$= ab^2a^{-1} = (aba^{-1})^2 = (b^2)^2 = b^4,$$

$$\text{so } c = b^{-1}(bc) = b^{-1}(b^4) = b^3.$$

So $b^6 = c^2 = bcb^{-1} = bb^3b^{-1} = b^3$. So $b^3 = 1$, so $c = 1$.

Then $a^2 = cac^{-1} = a$, so $a = 1$;

and then $b^2 = aba^{-1} = b$, so $b = 1$.

Consequently, $a = b = c = 1$, so $G = \{1\}$.