Universal covering spaces: A particularly important covering space of a space X to identify is one which is simply connected. Such a covering is essentially unique:

If X is *locally path-connected* and has two connected, simply connected covering spaces $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$, then since $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(\pi_1(X_2, x_2)) = \{1\} \subseteq \pi_1(X, x_0)$, the lifting criterion applied twice gives maps $\widetilde{p}_1 : (X_1, x_1) \to (X_2, x_2)$ and $\widetilde{p}_2 : (X_2, x_2) \to (X_1, x_1)$ with $p_2 \circ \widetilde{p}_1 = p_1$ and $p_1 \circ \widetilde{p}_2 = p_2$. Consequently, $p_2 \circ \widetilde{p}_1 \circ \widetilde{p}_2 = p_1 \circ \widetilde{p}_2 = p_2$ and similarly, $p_1 \circ \widetilde{p}_2 \circ \widetilde{p}_1 = p_2 \circ \widetilde{p}_1 = p_1$. So $\widetilde{p}_1 \circ \widetilde{p}_2 : (X_2, x_2) \to (X_2, x_2)$, for example, is a lift of p_2 to the covering map p_2. But so is the identity map! By uniqueness, therefore, $\widetilde{p}_1 \circ \widetilde{p}_2 = Id$. Similarly, $\widetilde{p}_2 \circ \widetilde{p}_1 = Id$. So (X_1, x_1) and (X_2, x_2) are homeomorphic. (More: there is a homeo interpolating between the covering maps.) So up to homeomorphism, a space has only one connected, simply-connected covering space. It is known as the *universal covering* of the space X.

Not every (locall path-connected) space X has a universal covering; a (further) necessary condition is that X be *semi-locally simply connected*. The idea is that If $p : \widetilde{X} \to X$ is the universal cover, then for every point $x \in X$, we have an evenly-covered neighborhood $\mathcal{U}$ of x. The inclusion $i : \mathcal{U} \to X$, by definition, lifts to $\widetilde{X}$, so $i_*(\pi_1(\mathcal{U}, x)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}) = \{1\}$, so i_* is the trivial map. Consequently, every loop in $\mathcal{U}$ is null-homotopic in X. This is semi-local simple connectivity; every point has a neighborhood whose inclusion-induced homomorphism is trivial. Not all spaces have this property; the most famous is the Hawaiian earrings $X = \bigcup_n \{x \in \mathbb{R}^2 : ||x - (1/n, 0)|| = 1/n\}$. The point $(0, 0)$ has no such neighborhood.

Building universal coverings: If a space X is path connected, locally path connected, and semi-locally simply connected (S-LSC), then it has a universal covering.

The idea is that a covering space has the path lifting and homotopy lifting properties, and the universal cover is the only covering space for which *only* null-homotopic loops lift to loops. So we build a space and a map which must have these properties. We do this by making a space $\widetilde{X}$ whose points are (equivalence classes $[\gamma]$ of) based paths $\gamma : (I, 0) \to (X, x_0)$, where two paths are equivalent if they are homotopic rel endpoints! The projection map is $p([\gamma]) = \gamma(1)$.

The S-LSCness of X guarantees that this is a covering map; choosing $x \in X$, a path γ_0 from x_0 to x, and a nbhd $\mathcal{U}$ of x guaranteed by S-LSC, a path θ from x_0 to points in $\mathcal{U}$ is homotopic to $\gamma * \gamma_0 * \eta$ where γ is a loop at x_0 and η is a path in $\mathcal{U}$. $[\gamma = \theta * \overline{\eta} * \overline{\gamma_0}.]$ But by S-LSC, a path in $\mathcal{U}$ is determined up to homotopy by its endpoints, and so, with γ fixed, these paths are in one-to-one correspondence with $\mathcal{U}$. So $p^{-1}(\mathcal{U})$ is a disjoint union, indexed by $\pi_1(X, x_0)$, of sets that are in 1-to-1 corresp with $\mathcal{U}$.

The appropriate topology on $\widetilde{X}$ is essentially given as a basis by triples $(\gamma, \gamma_0, \mathcal{U})$ as above. This topology makes p a covering map. Note that the inverse image of the basepoint x_0 is the equivalence classes of loops at x_0, i.e., $\pi_1(X, x_0)$. A path γ lifts to the path of paths γ_t, where $\gamma_t(s) = \gamma(ts)$, and so the only loops in X which lift to a loop in $\widetilde{X}$ have $[\gamma_1] = [\gamma] = [c_{x_0}]$, i.e., $[\gamma] = 1$ in $\pi_1(X, x_0)$. This implies that $p_*(\pi_1(\widetilde{X}, [c_{x_0}])) = \{1\}$, so $\pi_1(\widetilde{X}, [c_{x_0}]) = \{1\}$.

However, nobody in their right minds would go about building $\widetilde{X}$ in this way!

Why Care? The universal cover gives a unified approach to building all connected covering spaces of X. The key to this is the *deck transformation group (Deckbewegungsgruppe)* of a covering space $p : \widetilde{X} \to X$; this is the set of all homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $p \circ h = p$.

By def'n, these h permute each of the pt inverses of p. Since h is a lift of the projection map p, by the lifting criterion h is det'd by which point in $p^{-1}(x_0)$ it takes the basepoint $\widetilde{x}_0$ of $\widetilde{X}$ to. A deck transformation sending $\widetilde{x}_0$ to $\widetilde{x}_1$ exists $\Leftrightarrow p_*(\pi_1(\widetilde{X}, \widetilde{x}_0) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)$ [we need one inclusion to give h, and the opposite inclusion to ensure it is a bijection].

These two groups are *conjugate*, by the projection of a path from $\widetilde{x}_0$ to $\widetilde{x}_1$ (follow the change of basept iso down into $G = \pi_1(X, x_0)$). Paths in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$ are in 1-to-1 corresp with the cosets of $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$ in $p_*(\pi_1(X, x_0)$; so deck transformations are in 1-to-1 corresp with cosets whose representatives conjugate H to itself. The set of such elements in G is called the *normalizer of H in G*, and denoted $N_G(H)$ or simply $N(H)$. The deck transformation group is therefore in 1-to-1 correspondence with the group $N(H)/H$ under $h \mapsto$ the coset with representative the projection of the path from $\widetilde{x}_0$ to $h(\widetilde{x}_0)$. And since the lift h is essentially built by lifting paths, it follows quickly that this map is a homomorphism, hence an isomorphism.

Applying this to the universal covering space $p : \widetilde{X} \to X$, in this case $H = \{1\}$, so $N(H) = \pi_1(X, x_0)$. So the deck transformation group is isomorphic to $\pi_1(X, x_0)$. For example, this gives the quickest possible proof that $\pi_1(S^1) \cong \mathbb{Z}$, since $\mathbb{R}$ is a contractible covering space, whose deck transformations are the translations by integer distances.

Thus $\pi_1(X)$ acts on its universal cover as a group of homeomorphisms. And since this action is *simply transitive* on point inverses [there is exactly one (that's the simple part) deck transformation carrying any one point in a point inverse to any other one (that's the transitive part)], the quotient map from $\widetilde{X}$ to the orbits of this action is the projection map p. The evenly covered property of p implies that X does have the quotient topology under this action.

So every space it X the quotient of its universal cover (if it has one!) by its fundamental group $G = \pi_1(X, x_0)$, acting as the group of deck transformations. And the quotient map is the covering projection. So $X \cong \widetilde{X}/G$.

In general, a quotient of a space Z by a group action G need not be a covering map. The action must be *properly discontinuous*: for every point $z \in Z$, there is a neighborhood $\mathcal{U}$ of x so that $g \neq 1 \Rightarrow \mathcal{U} \cap g\mathcal{U} = \emptyset$. The evenly covered neighborhoods provide these for the universal cover. And conversely, the quotient of a space by a p.d. group action is a covering space.

But! Given $G = \pi_1(X, x_0)$ and its action on a univ cover $\widetilde{X}$, we can, instead of modding out by G, mod out by any <u>subgroup</u> H of G, to build $X_H = \widetilde{X}/H$. This is a space with $\pi_1(X_H) \cong H$, having $\widetilde{X}$ as univ covering. And since the quotient (covering) map $p_G : \widetilde{X} \to X = \widetilde{X}/G$ factors through $\widetilde{X}/H$, we have an induced map $p_H : \widetilde{X}/H \to X$, which is a covering map; open sets with trivial inclusion-induced homomorphism lift homeomorphically to $\widetilde{X}$, hence homeomorphically to $\widetilde{X}/H$; choosing lifts to each point inverse of $x \in X$ builds the evenly covering nbhds for p_H . So every subgroup of G is the fundamental group of a covering of X.

The Galois correspondence: Two coverings $p_1 : X_1 \to X$, $p_2 : X_2 \to X$ are *isomorphic* if there is a homeo $h : X_1 \to X_2$ with $p_1 = p_2 \circ h$. Choosing basepts x_1, x_2 mapping to $x_0 \in X$, then if $h(x_1) = x_2$, then $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(h_*(\pi_1(X_1, x_1))) = p_{2*}(\pi_1(X_2, x_2))$. If instead $h(x_1) = x_3$, then $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(\pi_1(X_2, x_3))$. But $\pi_1(X_2, x_2)$ and $\pi_1(X_2, x_3)$ are isomorphic, via a change of basept isomorphism $\widehat{\eta}$, where η is a path in X_2 from x_2 to x_3. Such a path projects to X as a loop at x_0, and since the change of basept isom is by "conjugating" by the path η, the resulting groups $p_{2*}(\pi_1(X_2, x_2))$ and $p_{2*}(\pi_1(X_2, x_3))$ are conjugate, by $[p_2 \circ \eta]$.

So choosing <u>any</u> basepts over x_0, isomorphic coverings give, under projection, conjugate subgroups of $\pi_1(X, x_0)$. But conversely, given covering spaces X_1, X_2 whose subgroups $p_{1*}(\pi_1(X_1, x_1))$ and $p_{2*}(\pi_1(X_2, x_2))$ are conjugate, lifting a loop γ representing the conjugating element to a loop $\widetilde{\gamma}$ in X_2 starting at x_2 gives, as its terminal endpoint, a point x_3 with $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(\pi_1(X_2, x_3))$ (since it conjugates back!), and so, by the lifting criterion, there is an isomorphism $h : (X_1, x_1) \to (X_2, x_3)$. So conjugate subgroups give isomorphic coverings. Thus:

The Galois correspondence: For a path-connected, locally path-connected, semi-locally simply-connected space X, the image of the induced homomorphism on π_1 gives a one-to-one correspondence between [isomorphism classes of (connected) coverings of X] and [conjugacy classes of subgroups of $\pi_1(X)$].

So, for example, if you have a group G that you are interested in, you know of a (nice enough) space X with $\pi_1(X) \cong G$, and you know enough about the coverings of X, then you can gain information about the subgroup structure of G.

For example, a free group $F(\Sigma)$ is π_1 of a bouquet of circles X. Any covering space $\widetilde{X}$ of X is a union of vertices and edges, so is a graph. Collapsing a maximal tree to a point, $\widetilde{X}$ is $\sim$ a bouquet of circles, so has free π_1. So every subgroup of a free group is free. A subgroup H of index n in $F(\Sigma)$ corresponds to a n-sheeted covering $\widetilde{X}$ of X. If $|\Sigma| = m$, then $\widetilde{X}$ will have n vertices and nm edges. Collapsing a maximal tree, having $n-1$ edges, to a point, leaves a bouquet of $nm - n + 1$ circles, so $H \cong F(nm - n + 1)$. For example, for $m = 3$, index n subgroups are free on $2n + 1$ generators, so every free subgroup on 4 generators has infinite index in $F(3)$. [Try proving that directly!]

Given a free group $G = F(a_1, \ldots a_n)$ and a collection of words $w_1, \ldots w_m \in G$, we can determine the rank and ndex of the subgroup it H they generate by building the corresponding cover. The idea is to start with a bouquet of m circles, each subdivided and labelled to spell out the words w_i. Then we repeatedly identify edges sharing on common vertex if they are labelled precisely the same (same letter *and* same orientation). This process is known as *folding*. One can inductively show that the (obvious) maps from these graphs to the bouquet of n circles X_n both have image H under the induced maps on π_1; since the map for the unfolded graph factors through the one for the folded graph, the image from the folded graph can only get smaller, but we can still spell out the same words as loops in the folded graph, so the image can also only have gotten bigger! We continue this folding process until there is no more folding to be done; the resulting graph X is what is known (in combinatorics) as a *graph covering*; the map to X_n is locally injective. If this map is a covering map, then our subgroup H has finite index (equal to the degree of the covering) and we can compute the rank of H (and a basis!) from the folded graph. If it is not a covering map, then the map is not locally surjective at some vertices; if we graft trees onto these vertices, we can extend the map to an (infinite-sheeted) covering map without changing the homotopy type of the graph. H therefore has infinite index in G, and its rank can be computed from $H \cong \pi_1(X)$.

Given words $w_1, \ldots , w_n \in F(x_1, \ldots x_m)$, we can build the covering space correpsonding to the subgroup $H = \langle w_1, \ldots , w_n \rangle$ by a process of *folding*, in so doing determining the index of H and a basis for H as a free group.

The idea is to build a covering $\widetilde{X}$ of the bouquet X_m of m circles, the image of whose fundamental group is H. Start with a bouquet Y of n circles, each subdivided and (orientedly) labeled to spell out the words w_i. This is a 1-complex; the labeling tells us how to map Y to X_m. Then inductively, we fold together any two edges at a vertex with the same oriented edge, since they are supposed to be mapping together in X_m, and that mapping will not give a local homeo! Note two things: folding is (almost) a homotopy equivalence, and the original words still always spell out loops in the intermediate folded spaces.

Stop when you run out of folds. The "obvious" map from the resulting space to X_m is locally injective, otherwise we have another fold to do. One of two things will occur at the end; either the map is everywhere a local homeo, and so is a covering map, or there are points where it is not locally surjective. In the first case, we have succeeded in building a finite covering $\widetilde{X}$ with (since the w_i still generate the fundamental group) fundamental group having image H, and we can read off the index of and a basis for H from the covering. In the second case, we can extend our space $\widetilde{X}$ to a covering by grafting on (infinite) trees, so H has infinite index; since the grafted space deformation retracts to $\widetilde{X}$, we can still read off a basis for H by the same process.