Now all we need are some new chain complexes!

Relative homology: Start with a pair (X, A) , i.e., of a space X and a subspace $A \subseteq X$. As abelian groups we can think of $C_n(A)$ as a subgroup of $C_n(X)$ (under the injective homom induced by the inclusion $i : A \to X$), and we can set $C_n(X, A) = C_n(X)/C_n(A)$. The boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ satisfies $\partial_n(C_n(A) \subseteq C_{n-1}(A)$ (the boundary of a map into A maps into A), so we get an induced boundary map $\partial_n : C_n(X, A) \to C_{n-1}(X, A)$. The $(C_n(X, A), \partial_n)$ are a chain complex; its homology groups are the *singular relative homology groups of the pair* (X, A), denoted $H_n(X, A)$. A cycle is $[z]$ with $\partial z \in C_{n-1}(A)$, i.e., a chain with boundary in A. A boundary satisfies $z = \partial w + a$ for some $w \in C_{n+1}(X)$ and $a \in C_n(A)$, i.e., it *cobounds* a chain in A ($\partial w = z - a$). Note that the relative homology of the pair $(X, \emptyset)$ is just the ordinary homology of X; we quotient out by nothing.

There is a reduced relative homology as well, since we can augment with the same map (1-simplices always have 2 ends!), but in this case it has (essentially) no effect; $\widetilde{H}_i(X, A) \cong H_i(X, A)$ for all i <u>unless</u> $A = \emptyset$, in which case we lose the $\mathbb{Z}$ in dimension 0 that we expect to.

The inclusion i_n and projection p_n maps give us short exact sequences

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0$$

and since the boundary on chains in X restricts to the boundary on A and induces the boundary on (X, A), i_n and p_n are chain maps. So we get a long exact homology sequence

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X, A) \to H_{n-1}(A) \to H_{n-1}(X) \to \cdots$$

There is also a long exact sequence of a triple (X, A, B), where by triple we mean $B \subseteq A \subseteq X$. From the short exact sequences

$$0 \to C_n(A, B) \to C_n(X, B) \to C_n(X, A) \to 0 \qquad \text{, i.e.,}$$
$$0 \to C_n(A)/C_n(B) \to C_n(X)/C_n(B) \to C_n(X)/C_n(A) \to 0$$

we get the long exact sequence

$$\cdots \to H_n(A, B) \to H_n(X, B) \to H_n(X, A) \to H_{n-1}(A, B) \to H_{n-1}(X, B) \to \cdots$$

A map of pairs $f : (X, A) \to (Y, B)$ induces (by postcomposition) a homom of relative homology $f_* : H_i(X, A) \to H_i(Y, B)$, just as with absolute homology. We also get a homotopy-invariance result: if $f, g : (X, A) \to (Y, B)$ are maps of pairs which are *homotopic as maps of pairs*, i.e., there is a map $(X \times I, A \times I) \to (Y, B)$ which is f on one end and g on the other, then $f_* = g_*$. The proof is identical to the one given before; the prism map P sends chains in A to chains in A, so induces a map $C_i(X \times I, A \times I) \to C_{i+1}(X, A)$ which does precisely what we want.

There is one other main piece of homological algebra that we will find useful ; the **Five Lemma**. Now that we have a way of building long exact sequences, we will soon have ways of building maps between them. So the next result becomes useful.

If we have abelian groups and homoms, giving two exact sequences

$$\begin{array}{ccccccccc} A_n & \stackrel{f_n}{\to} & B_n & \stackrel{g_n}{\to} & C_n & \stackrel{h_n}{\to} & D_n & \stackrel{i_n}{\to} & E_n \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \epsilon\downarrow \\ A_{n-1} & \stackrel{f_{n-1}}{\to} & B_{n-1} & \stackrel{g_{n-1}}{\to} & C_{n-1} & \stackrel{h_{n-1}}{\to} & D_{n-1} & \stackrel{i_{n-1}}{\to} & E_{n-1} \end{array}$$

and the homoms $\alpha, \beta, \delta, \epsilon$ are all isomorphisms, then γ is an isomorphism.

The proof is literally a matter of doing the only thing you can. To show injectivity, suppose $x \in C_n$ and $\gamma x = 0$, then $h_{n-1}\gamma x = \delta h_n x = 0$, so, since δ is injective, $h_n x = 0$. So by the exactness at C_n, $x = g_n y$ for some $y \in B_n$. Then $g_{n-1}\beta y = \gamma g_n y = \gamma x = 0$, so by exactness at B_{n-1}, $\beta y = f_{n-1} z$ for some $z \in A_{n-1}$. Then since α is surjective, $f_{n-1} z = \alpha w$ for some w. Then $0 = g_n f_n w$. But $\beta f_n w = f_{n-1}\alpha w$ $f_{n-1} z = \beta y$, so since β is injective, $y = f_n w$. So $0 = g_n f_n w = g_n y = x$. So $x = 0$.

For surjectivity, suppose $x \in C_{n-1}$. Then $h_{n-1} x \in D_{n-1}$, so since δ is surjective, $h_{n-1} x = \delta y$ for some $y \in D_n$. Then $\epsilon i_n y = i_{n-1}\delta y = i_{n-1} h_{n-1} x = 0$, so since ϵ is injective, $i_n y = 0$. So by exactness at D_n, $y = h_n z$ for some $z \in C_n$. Then $h_{n-1}\gamma z = \delta h_n z = \delta y = h_{n-1} x$, so $h_{n-1}(\gamma z - x) = 0$, so by exactness at C_{n-1}, $\gamma z - x = g_{n-1} w$ for some $w \in B_{n-1}$. Then since β is surjective, $w = \beta u$ for some $u \in B_n$. Then $\gamma g_n u = g_{n-1}\beta u = g_{n-1} w = \gamma z - x$, so $x = \gamma z - \gamma g_n u = \gamma(z - g_n u)$. So γ is onto.