Homology on "small" chains = singular homology: The point to all of these calculations was that if $\{\mathcal{U}_\alpha\}$ is an open cover of X, then the inclusions $i_n : C_n^{\mathcal{U}}(X) \to C_n(X)$ induce isomorphisms on homology. This gives us two big theorems. First:

Mayer-Vietoris Sequence: If $X = \mathcal{U} \cup \mathcal{V}$ is the union of two open sets, then the short exact sequences $0 \to C_n(\mathcal{U} \cap \mathcal{V}) \to C_n(\mathcal{U}) \oplus C_n(\mathcal{V}) \to C_n^{\{\mathcal{U},\mathcal{V}\}}(X) \to 0$, together with the isomorphism above, give the long exact sequence

$$\cdots \to H_n(\mathcal{U} \cap \mathcal{V}) \stackrel{(i_{\mathcal{U}*}, -i_{\mathcal{V}*})}{\to} H_n(\mathcal{U}) \oplus H_n(\mathcal{V}) \stackrel{j_{\mathcal{U}*}+j_{\mathcal{V}*}}{\to} H_n(X) \stackrel{\partial}{\to} H_{n-1}(\mathcal{U} \cap \mathcal{V}) \to \cdots$$

As with Seifert - van Kampen, we can replace open sets by sets A, B having nbhds $\mathcal{U}, \mathcal{V}$ which def. retract to them, so that $\mathcal{U} \cap \mathcal{V}$ def. retracts to $A \cap B$. E.g., subcomplexes $A, B \subseteq X$ of a CW-complex, with $A \cup B = X$ have homology satisfying a l.e.s.

$$\cdots \to H_n(A \cap B) \stackrel{(i_{A*}, -i_{B*})}{\to} H_n(A) \oplus H_n(B) \stackrel{j_{A*}+j_{B*}}{\to} H_n(X) \stackrel{\partial}{\to} H_{n-1}(A \cap B) \to \cdots$$

For reduced homology, we augment the chain complexes used above with the s.e.s. $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \to 0$, where the maps are $a \mapsto (a, -a)$ and $(a, b) \mapsto a + b$.

E.g., an n-sphere S^n is the union $S^n_+ \cup S^n_-$ of its upper and lower hemispheres, each of which is contractible, and have intersection $S^n_+ \cap S^n_- = S^{n-1}_0$ the equatorial $(n-1)$-sphere. So Mayer-Vietoris gives us the exact sequence

$\cdots \to \widetilde{H}_k(S^n_+) \oplus \widetilde{H}_k(S^n_-) \to \widetilde{H}_k(S^n) \to \widetilde{H}_{k-1}(S^{n-1}_0) \to \widetilde{H}_{k-1}(S^n_+) \oplus \widetilde{H}_{k-1}(S^n_-) \to \cdots$, i.e,

$0 \to \widetilde{H}_k(S^n) \to \widetilde{H}_{k-1}(S^{n-1}_0) \to 0$ i.e., $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$ for every k and n. So by induction,

$$\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-n}(S^0) \cong \begin{cases} \mathbb{Z}, & \text{if k=n} \\ 0, & \text{otherwise} \end{cases}$$

The second result that this machinery gives us is what is properly known as *excision*:

If $B \subseteq A \subseteq X$ and $\mathrm{cl}_X(B) \subseteq \mathrm{int}_X(A)$, then for every k the inclusion-induced map $H_k(X \setminus B, A \setminus B) \to H_k(X, A)$ is an isomorphism.

An equivalent formulation of this is that if $A, B \subseteq X$ and $\mathrm{int}_X(A) \cup \mathrm{int}_X(B) = X$, then the inclusion-induced map $H_k(B, A \cap B) \to H_k(X, A)$ is an isomorphism. [From first to second statement, set $B' = X \setminus B$.]

To prove the second statement, we know that the inclusions $C_n^{\{A,B\}}(X) \to C_n(X)$ induce isomorphisms on homology, as does $C_n(A) \to C_n(A)$, so, by the five lemma, the induced map
$$C_n^{\{A,B\}}(X)/C_n(A) \to C_n(X)/C_n(A) = C_n(X, A)$$
induces isomorphisms on homology. But the inclusion
$$C_n(B) \to C_n^{\{A,B\}}(X)$$
induces a map
$$C_n(B, A \cap B) = C_n(B)/C_n(A \cap B) \to C_n^{\{A,B\}}(X)/C_n(A)$$
which is an isomorphism of chain groups; a basis for $C_n^{\{A,B\}}(X)/C_n(A)$ consists of singular simplices which map into A or B, but don't map into A, i.e., of simplices mapping into B but not A, i.e., of simplices mapping into B but not $A \cap B$. But this is the <u>same</u> as the basis for $C_n(B, A \cap B)$!

With these tools, we can start making some <u>real</u> homology computations. First, we show that if $\emptyset \neq A \subseteq X$ is "nice enough", then $H_n(X, A) \cong \widetilde{H}_n(X/A)$. The definition of nice enough, like Seifert - van Kampen, is that A is closed and has an open neighborhood $\mathcal{U}$ that deformation retracts to A (think: A is the subcomplex of a CW-complex X). Then using $\mathcal{U}, X \setminus A$ as a cover of X, and $\mathcal{U}/A, (X \setminus A)/A$ as a cover of X/A, we have

$$\widetilde{H}_n(X/A) \overset{(1)}{\cong} H_n(X/A, A/A) \overset{(2)}{\cong} H_n(X/A, \mathcal{U}/A) \overset{(3)}{\cong} H_n(X/A \setminus A/A, \mathcal{U}/A \setminus A/A) \overset{(4)}{\cong} H_n(X \setminus A, \mathcal{U} \setminus A) \overset{(5)}{\cong} H_n(X, A)$$

Where (1),(2) follow from the LES for a pair, (3),(5) by excision, and (4) because the restriction of the quotient map $X \to X/A$ gives a homeomorphism of pairs.

Second, if X, Y are T_1, $x \in X$ and $y \in Y$ each have neighborhoods $\mathcal{U}, \mathcal{V}$ which deformation retract to each point, then the one-point union $Z = X \vee Y = (X \coprod Y)/(x = y)$ has $\widetilde{H}_n(Z) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$; this follows from a similar sequence of isomorphisms. Setting $z =$ the image of $\{x, y\}$ in Z, we have

$$\widetilde{H}_n(Z) \cong H_n(Z, z) \cong H_n(Z, \mathcal{U} \vee \mathcal{V}) \cong H_n(Z \setminus z, \mathcal{U} \vee \mathcal{V} \setminus z) \cong H_n([X \setminus x] \coprod [Y \setminus y], [\mathcal{U} \setminus x] \coprod [\mathcal{V} \setminus y]) \cong H_n(X \setminus x, \mathcal{U} \setminus x) \oplus H_n(Y \setminus y, \mathcal{V} \setminus y) \cong H_n(X, x) \oplus H_n(Y, y) \cong \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y)$$

By induction, we then have $\widetilde{H}_n(\vee_{i=1}^k X_i) \cong \oplus_{i=1}^k \widetilde{H}_n(X_i)$