Simplicial homology = singular homology: We have so far introduced two homologies; simplicial, H_*^Δ, whose computation "only" required some linear algebra, and singular, H_*, which is formally less difficult to work with, and which, you may suspect by now, is also becoming less difficult to compute. For Δ-complexes, these homology groups are the same, $H_n^\Delta(X) \cong H_n(X)$ for every X. In fact, the isomorphism is induced by the inclusion $C_n^\Delta(X) \subseteq C_n(X)$. We almost have the tools necessary to prove this; we need to note that most of the edifice we have built for singular homology could have been built for simplicial homology, including relative homology (for a sub-Δ-complex A of X), and a SES of chain groups, giving a LES sequence for the pair,

$$\cdots \to H_n^\Delta(A) \to H_n^\Delta(X) \to H_n^\Delta(X,A) \to H_{n-1}^\Delta(A) \to \cdots$$

The proof of the isomorphism between the two homologies proceeds by first showing that the inclusion induces an isomorphism on k-skeleta, $H_n^\Delta(X^{(k)}) \cong H_n(X^{(k)})$, by induction on k using the Five Lemma applied to the diagram

$$\begin{array}{ccccccccc} H_{n+1}^\Delta(X^{(k)},X^{(k-1)}) & \to & H_n^\Delta(X^{(k-1)}) & \to & H_n^\Delta(X^{(k)}) & \to & H_n^\Delta(X^{(k)},X^{(k-1)}) & \to & H_{n-1}^\Delta(X^{(k-1)}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{n+1}(X^{(k)},X^{(k-1)}) & \to & H_n(X^{(k-1)}) & \to & H_n(X^{(k)}) & \to & H_n(X^{(k)},X^{(k-1)}) & \to & H_{n-1}(X^{(k-1)}) \end{array}$$

The second and fifth vertical arrows are, by an inductive hypothesis, isomorphisms.

$$\begin{array}{ccccccccc} H^\Delta_{n+1}(X^{(k)},X^{(k-1)}) & \to & H^\Delta_n(X^{(k-1)}) & \to & H^\Delta_n(X^{(k)}) & \to & H^\Delta_n(X^{(k)},X^{(k-1)}) & \to & H^\Delta_{n-1}(X^{(k-1)}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{n+1}(X^{(k)},X^{(k-1)}) & \to & H_n(X^{(k-1)}) & \to & H_n(X^{(k)}) & \to & H_n(X^{(k)},X^{(k-1)}) & \to & H_{n-1}(X^{(k-1)}) \end{array}$$

The first and fourth vertical arrows are isomorphisms because, essentially, we can, in each case, identify these groups. $H_n(X^{(k)},X^{(k-1)}) \cong H_n(X^{(k)}/X^{(k-1)}) \cong \widetilde{H}_n(\vee S^k)$ are either 0 (for $n \neq k$) or $\oplus\mathbb{Z}$ (for $n = k$), one summand for each n-simplex in X. But the same is true for $H^\Delta_n(X^{(k)},X^{(k-1)})$ (the chain groupsare 0 for $n \neq k$); and for $n = k$ the generators are precisely the n-simplices of X. The inclusion-induced map takes generators to generators, so is an isomorphism. So by the Five Lemma, the middle rows are also isomorphisms, completing our inductive proof.

Returning to $H^\Delta_n(X)\xrightarrow{I_*}H_n(X)$, we show that this map is an isomorphism. Any $[z] \in H_n(X)$ is rep'd by a cycle $z = \sum a_i\sigma_i$ for $\sigma_i : \Delta^n \to X$. But each image $\sigma_i(\Delta^n)$ is compact, and so meets only finitely-many cells of X. So there is a k for which all of the simplices map into $X^{(k)}$, and so we may treat $z \in C_n(X^{(k)}$. Viewed this way, it is still a cycle, and so $[z] \in H_n(X^{(k)}) \cong H^\Delta_n(X^{(k)})$ so there is a $z' \in C^\Delta_n(X^{(k)})$ and a $w \in C_{n+1}(X^{(k)})$ with $i_\# z' - z = \partial w$. But thinking of $z' \in C^\Delta_n(X)$ and $w \in C_{n+1}(X)$, we have the same equality, so $[z'] \in H^\Delta_n(X)$ and $i_*[z'] = [z]$. So i_* is surjective. If $i_*([z]) = 0$, then the cycle $z = \sum a_i\sigma_i$ is a sum of characteristic maps of n-simplices of X, and so can be thought of as an element of $C^\Delta_n(X^{(n)})$. Being 0 in $H_n(X)$, $z = \partial w$ for some $w \in C_{n+1}(X)$. But as before, $w \in C_n(X^{(r)})$ for some r, and so thought of as an element of the image of the isomorphism $i_* : H^\Delta_n(X^{(r)}) \to H_n(X^{(r)})$, $i_*([z]) = 0$, so $[z] = 0$. So $z = \partial u$ for some $u \in C^\Delta_{n+1}(X^{(r)}) \subseteq C^\Delta_{n+1}(X)$. So $[z] = 0$ in $H^\Delta_n(X)$. Consequently, simplicial and singular homology groups are isomorphic.

The isomorphism between simplicial and singular homology provides very quick proofs of several results about singular homology, which would other would require some effort:

If the Δ-complex X has no simplices in dimension greater than n, then $H_i(X) = 0$ for all $i > n$.

This is because the simplicial chain groups $C_i^\Delta(X)$ are 0, so $H_i^\Delta(X) = 0$.

If for each n, the Δ-complex X has finitely many n-simplices, then $H_n(X)$ is finitely generated for every n.

This is because the simplicial chain groups $C_n^\Delta(X)$ are all finitely generated, so $H_n^\Delta(X)$, being a quotient of a subgroup, is also finitely generated. [We are using here that the number of generators of a subgroup H of an *abelian* group G is no larger than that for G; this is not true for groups in general!]

A quick Mayer-Vietoris computation allows us to compute the homology groups of surfaces: Σ_g = a 2-disk D glued to a bouquet X of $2g$ circles, with "intersection" a circle, so we have

$\widetilde{H}_2(X) \oplus \widetilde{H}_2(D) \to \widetilde{H}_2(\Sigma_g) \to \widetilde{H}_1(S^1) \to \widetilde{H}_1(X) \oplus \widetilde{H}_1(D) \to \widetilde{H}_1(\Sigma_g) \to \widetilde{H}_0(S^1)$

i.e., $0 \oplus 0 \to \widetilde{H}_2(\Sigma_g) \xrightarrow{\partial} \mathbb{Z} \to \mathbb{Z}^{2g} \oplus 0 \to \widetilde{H}_1(\Sigma_g) \to 0$

But the map $\mathbb{Z} \to \mathbb{Z}^{2g}$ is 0; the generator of $H_1(S^1)$ is taken to the sum of the edges in the identifcation map, which cancel in pairs. So we really have the SES's

$0 \to \widetilde{H}_2(\Sigma_g) \xrightarrow{\partial} \mathbb{Z} \to 0$ and $0 \to \mathbb{Z}^{2g} \to \widetilde{H}_1(\Sigma_g) \to 0$, so $\widetilde{H}_2(\Sigma_g) = \mathbb{Z}$ and $\widetilde{H}_1(\Sigma_g) = \mathbb{Z}^{2g}$; all others are 0 by dimension and connectedness considerations.

Some more topological results with homological proofs: The Klein bottle and real projective plane cannot embed in $\mathbb{R}^3$. This is because a surface Σ embedded in $\mathbb{R}^3$ has a (the proper word is *normal*) neighborhood $N(\Sigma)$, which deformation retracts to Σ; literally, it is all points within a (uniformly) short distance in the normal direction from the point on the surface Σ. Our non-embeddedness result follows (by contradiction) from applying Mayer-Vietoris to the pair $(A, B) = (\overline{N(\Sigma)}, \overline{\mathbb{R}^3 \setminus N(\Sigma)})$, whose intersection is the boundary $F = \partial N(\Sigma)$ of the normal neighborhood. The point, though, is that F is an orientable surface; the outward normal (pointing away from $N(\Sigma)$) at every point, taken as the first vector of a right-handed orientation of $\mathbb{R}^3$ allows us to use the other two vectors as an orientation of the surface. So F is one of the surface F_g above whose homologies we just computed. This gives the LES $\widetilde{H}_2(\mathbb{R}^3) \to \widetilde{H}_1(F) \to \widetilde{H}_1(A) \oplus \widetilde{H}_1(B) \to \widetilde{H}_1(\mathbb{R}^3)$ which renders as $0 \to \mathbb{Z}^{2g} \to \widetilde{H}_1(\Sigma) \oplus G \to 0$, i.e., $\mathbb{Z}^{2g} \cong \widetilde{H}_1(\Sigma) \oplus G$. But for the Klein bottle and projective plane (or any closed, non-orientable surface for that matter), $\widetilde{H}_1(\Sigma)$ has torsion, so it cannot be the direct summand of a torsion-free group! So no such embedding exists. This result holds more generally for any 2-complex K whose (it turns out it would have to be first) homology has torsion; any embedding into $\mathbb{R}^3$ would have a neighborhood deformation retracting to K, with boundary a (for the exact same reasons as above) closed orientable surface.