Cohomology: There is "dual" theory to the homology theory, called "cohomology theory", which is based on the following observation: if we have a chain complex

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_0 \to 0$$

with, we assume for the moment, finitely-generated free abelian chain groups, then, thinking in terms of $\mathbb{Z}$-vector spaces, the boundary maps are represented by integer matrices $d_n : C_n \to C_{n-1}$. The *transposes* of these matrices give rise to linear maps $d_n^{\mathrm{T}} : C_{n-1} \to C_n$, which form a chain complex (since $d_n^{\mathrm{T}} d_{n-1}^{\mathrm{T}} = (d_{n-1} d_n)^{\mathrm{T}} = 0^{\mathrm{T}} = 0$) *running the opposite direction*

$$0 \to C_0 \to C_1 \to \cdots \to C_{n-1} \xrightarrow{\delta_n} C_n \to \cdots$$

whose homology groups we can then compute in the usual way. This is the basic idea behind cohomology. The transpose is not "really" a map from C_{n-1} to C_n, though, it is more properly thought of as a map between the dual vector spaces C_{n-1}^*, C_n^*. The formal way to construct the theory is to introduce the "Hom" functor: for a (fixed) abelian group G and an abelian group A,

$$\mathrm{Hom}(A, G) = \{f : A \to G : f \text{ is a homomomorphism}\} .$$

For $\mathbb{R}$-vector spaces V, for example, the usual dual vector space is $V^* = \mathrm{Hom}(V, \mathbb{R})$. The basic point is that Hom "turns arrows around"; given a homomorphism $\varphi : A \to B$, there is an induced homomorphism $\varphi^* : \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$ given by $\varphi^*(f) = f \circ \varphi$. This satisfies $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$ and $I^* = I$, as a quick computation establishes. Just as important for us is that $0^* = 0$.

Given a chain complex $\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots \to C_0 \to 0$ (with no assumptions on the chain groups) and a "coefficient" group G, we can dualize the complex to obtain

$$(*) \qquad 0 \to \mathrm{Hom}(C_0, G) \to \mathrm{Hom}(C_1, G) \to \cdots \to \mathrm{Hom}(C_{n-1}, G) \xrightarrow{\partial_n^*} \mathrm{Hom}(C_n, G) \to \cdots$$

We will usually write $\partial_n^* = \delta_n$, calling them the *coboundary* maps. These coboundary maps are defined by $\partial_n^*(f)(x) = f(\partial_n x)$ for $f \in \mathrm{Hom}(C_{n-1}, G)$ and $x \in C_n$ (since $\partial_n^*(f) \in \mathrm{Hom}(C_n, G)$). Just as in the finitely generated case, $\delta_{n+1} \circ \delta_n = 0$, since for any $f \in \mathrm{Hom}(C_{n-1}, G)$, and $x \in C_{n+1}$,

$$\delta_{n+1} \circ \delta_n(f)(x) = \delta_n(f)(\partial_{n+1} x) = f(\partial_n \partial_{n+1} x)\ f(0) = 0,$$

so $\delta_{n+1} \circ \delta_n(f) = 0$. Consequently, $(*)$ is a chain complex (although technically, with its indices rising it is called a *cochain complex*), with the attendant homology groups (which we call *cohomology groups*!). We could treat this, really, as homology, by renumbering so that $C'_{-n} = \mathrm{Hom}(C_n, G)$; then the coboundary map decreases index, although our homology groups then live in negative dimensions! But cohomology is inherently arrow-reversing, so it seems better to just live with index-increasing maps?

In particular, starting with a space (or Δ-complex or CW-complex) X, dualizing the standard singular (or simplicial or cellular) chain complex $C_n(X)$ we obtain the singular (or...) cochain complex $(C^n(X; G), \delta_{n-1})$, whose elements are *cochains*, and whose homology groups are the singular (or...) cohomology groups of X, with coefficients in G. As with homology, it is not immediately clear that the simplicial and cellular cohomology groups are topological invariants, but the singular homology groups are; the only input is X, from which we build $C_n(X)$ and $C^n(X) = \mathrm{Hom}(C_n(X), G)$.

We shall see that much of the edifice that we have built around singular homology goes through, with small changes made necessary by the reversal of arrows. One way to see a large part of this is by the fact that the homology groups of a chain complex determine the associated cohomology groups. To see how this might be formulated, we first note that there is a homomorphism $h : H^n(\mathcal{C};G) \to \mathrm{Hom}(H_n(\mathcal{C}),G)$, for any chain complex $\mathcal{C}$, defined as follows: given $[f] \in H^n(\mathcal{C};G)$ and $[z] \in H_n(\mathcal{C})$, we have $f \in \mathrm{Hom}(C_n, G)$ (with $\delta f = 0$; it is a *cocycle*), and $z \in Z_n \subseteq C_n$, so the element $f(z) \in G$ makes sense. So why not just try $h([f])([z]) = f(z)$? We can show that this is well-defined; if $[z] = [z']$, then $[z] - [z'] = [z - z'] = 0$, so $z - z' = \partial w$ for some $w \in C_{n+1}$, and then $f(z) - f(z') = f(z - z') = f(\partial w) = (\delta f)(w) = 0$, since $\delta f = 0$. OTOH, if $[f] = [f']$, then $f - f' = \delta g$ for some $g \in \mathrm{Hom}(C_{n-1}, G)$, and then $f(z) - f'(z) = \delta g(z) = g(\partial z) = g(0) = 0$, since z is an n-cycle. So h is well-defined. And since $h([f] - [f'])([z]) = (f - f')(z) = f(z) - f'(z) = h([f])([z]) - h([f'])([z])$, we have $h([f] - [f']) = h([f]) - h([f'])$, so h is a homomorphism.

Even more, though, if the chain groups C_n are free abelian, then h is onto. To see this, note that any $\varphi : H_n\mathcal{C} = Z_n/B_n \to G$ gives rise to a homom $\varphi_1 : Z_n \to G$, by $\varphi_1(z) = \varphi([z])$. But since C_n is free abelian, $Z_n = \ker \partial_n$ is a direct summand of C_n; $B_{n-1} = \mathrm{im}\, \partial_n \subseteq C_{n-1}$ is a subgroup of a free abelian group, so is free abelian, and a basis for B_{n-1}, pulled back to a collection of elements $\{v_i\}$ of C_n, together with a basis for Z_n, gives a basis for C_n. [Showing that the two bases span complementary subspaces, and together span C_n, is straightforward.] The point to this is that our homom φ_1 can be extended to a homom $\varphi_2 : C_n \to G$ by declaring that $\varphi_2(v_i) = 0$ for all i and that $\varphi_2 = \varphi_1$ on Z_n. Then $\delta(\varphi_2) = 0$, since $\delta(\varphi_2)(x) = \varphi_2(\partial x) = \varphi_1(\partial x) = \varphi([\partial x]) = \varphi(0) = 0$ for all x. [$\varphi_2(\partial x) = \varphi_1(\partial x)$ since $\partial x \in B_n \subseteq Z_n$.]

So φ_2 is a cocycle, and $h([\varphi_2])([z]) = \varphi_2(z) = \varphi_1(z) = \varphi([z])$, so $h([\varphi_2]) = \varphi$, as desired.
So we have the beginnings of a short exact sequence;

$$0 \to \ker h \to H^n(\mathcal{C}; G) \xrightarrow{h} \mathrm{Hom}(H_n(\mathcal{C}), G) \to 0$$

This sequence splits; our construction above actually describes a homom $k : \varphi \mapsto \varphi_2$, since $\varphi \mapsto \varphi_1$ is a homom, and φ_2 is essentially φ_1 extended by 0 to a subspace complementary to Z_n (in matrix terms, we pad the matrix for φ_1 with columns of 0's). Since $h(\varphi_2) = \varphi$, k is a right inverse to h. This then implies, by the Splitting Lemma (?), that

$$H^n(\mathcal{C}; G) \cong \mathrm{Hom}(H_n(\mathcal{C}), G) \oplus \ker h$$

and so to show that cohomology depends only on the homology groups of $\mathcal{C}$, it remains to show that $\ker h$ can be computed from the homology groups.

$$\ker h = \{[f] \ : \ f : C_n \to G \text{ and } f(z) = 0 \text{ for all } z \in C_n \text{ with } \partial z = 0\}$$

There is another map $j : \mathrm{Hom}(B_{n-1}, G) \to H^n(\mathcal{C}; G)$ given by $j(\varphi) = [\psi]$, where $\psi : C_n \to G$ is defined by $\psi(x) = \varphi(\partial x)$ [note that $\delta\psi(x) = \psi(\partial x) = \varphi(\partial^2 x) = 0$ for all x, so ψ is a cocycle], and again, this map is a homomorphism. Further, $\mathrm{im}\ j = \ker h$, since $h(j(\varphi))([z]) = h([\psi])([z]) = \psi(z) = \varphi(\partial z) = \varphi(0) = 0$ for all $[z]$, giving one containment, and given ψ with $h(\psi) = 0$, we <u>define</u> $\varphi : B_{n-1} \to G$ by $\varphi(\partial x) = \psi(x)$; if $\partial x = \partial y$, then $\psi(x - y) = 0$ since $\partial(x - y) = 0$, so $\psi(x) = \psi(y)$, and so φ is well-defined. Yet again, φ is a homom. And certainly $j(\varphi) = \psi$ (by pretending that the equation $\varphi(\partial x) = \psi(x)$ defines ψ !).

$$j(\varphi) = [\psi : x \mapsto \varphi(\partial x)] \in H^n(\mathcal{C}, G)\}$$

Therefore, by one of the isomorphism theorems, $\ker h \cong \text{Hom}(B_{n-1}, G)/\ker j$. But $\ker j$ consists of those maps $\varphi : B_{n-1} \to G$ for which $x \mapsto \varphi(\partial x)$ is a coboundary $(\delta\psi)(x) = \psi(\partial x)$ for some $\psi : C_{n-1} \to G$. On the face of it, it looks like φ itself could stand in for ψ, but the point is that φ and ψ have different domains. ψ has domain C_{n-1}, while φ has domain B_{n-1}. But this means that $\ker j$ is the image of the map $\text{Hom}(C_{n-1}, G) \to \text{Hom}(B_{n-1}, G)$ dual to the inclusion map $B_{n-1} \hookrightarrow C_{n-1}$. Note that we can put a third term in the middle of these two;

$$\text{Hom}(C_{n-1}, G) \to \text{Hom}(Z_{n-1}, G) \to \text{Hom}(B_{n-1}, G)$$

since $B_{n-1} \hookrightarrow Z_{n-1} \hookrightarrow C_{n-1}$. But the map $\text{Hom}(C_{n-1}, G) \to \text{Hom}(Z_{n-1}, G)$ is surjective, since Z_{n-1} is a direct summand of C_{n-1} (as before, we extend a map φ from z_{n-1} by zero of a complementary subspace to build a map from C_{n-1} whose image is φ). So $\ker j$ is also the image of the dual to the inclusion $i : B_{n-1} \hookrightarrow Z_{n-1}$.

The reason for tinkering with things in this way is that B_{n-1} and Z_{n-1} fit into a short exact sequence

$$(**)\ 0 \to B_{n-1} \to Z_{n-1} \to H_{n-1}(\mathcal{C}) \to 0$$

with dual sequence

$$(***)\ 0 \leftarrow \text{Hom}(B_{n-1}, G) \leftarrow \text{Hom}(Z_{n-1}, G) \leftarrow \text{Hom}(H_{n-1}(\mathcal{C}), G) \leftarrow 0$$

This sequence is not exact, but it is a cochain complex, and so has its own homology groups. Note that the group that we are after is the homology of this cochain complex at the spot $\text{Hom}(B_{n-1}, G)$. Since by hypothesis C_{n-1} is free abelian, so are B_{n-1} and Z_{n-1}; (**) is then an example of a *free resolution* of the abelian group $H_{n-1}(\mathcal{C})$.

Since I am getting tired of doing homological algebra and not topology, we will finish our proof that $\ker h$ (which we now know to be the (co)homology group mentioned above) depends only on the homology of $\mathcal{C}$ by appealing to:

The homology groups of the cochain complex dual to a free resolution of a group H depend only on the group H, and not on the particular free resolution chosen.

The particular homology group that we are interested in is known in the literature a $\text{Ext}(H, G)$. We will not be interested in knowing why it is called that, but only in the fact that, as a consequence we have the

Universal coefficients Theorem: For any chain complex $\mathcal{C}$,
$H^n(\mathcal{C}; G) \cong \text{Hom}(H_n(\mathcal{C}), G) \oplus \text{Ext}(H_{n-1}(\mathcal{C}), G)$.

together with some observations on how to calculate Ext, based on its indifference to the resolution used to compute it. From the exact sequence

$0 \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$, the dual $0 \leftarrow 0 \leftarrow G \leftarrow G \leftarrow 0$ gives $\text{Ext}(\mathbb{Z}, G) = 0$; from

$0 \to \mathbb{Z} \overset{\times n}{\to} \mathbb{Z} \to \mathbb{Z}_n \to 0$, the dual

$0 \leftarrow G \overset{\times n}{\leftarrow} G \to \text{Hom}(\mathbb{Z}_n, G) \to 0$ gives $\text{Ext}(\mathbb{Z}_n, G) = G/nG$; and the fact that

$\text{Ext}(H_1 \oplus H_2, G) \cong \text{Ext}(H_1, G) \oplus \text{Ext}(H_2, G)$ (by taking the direct sum of two resolutions, and using the fact that $\text{Hom}(-, G)$ respects direct products, and that homology respects direct products)

suffice to compute $\text{Ext}(H, G)$ for any finitely-generated abelian group H, which will usually suffice for our purposes.

Applying all of this homological algebra (there is no topology underlying any of the above work, except as motivation) to our chain complexes from a space X, we find that

$$H^n(X; G) \cong \mathrm{Hom}(H_n(X), G) \oplus \mathrm{Ext}(H_{n-1}(X), G) \ .$$

Since the groups on the right are the same whether we use singular, simplicial, or cellular chain complexes to build them, the same is true for the left. So the singular, simplicial, and cellular cohomology groups are all isomorphic (when any two of them are defined)! If we were to chase through the computations above, we could recover the fact that the isomorphisms can be induced by the inclusion maps of the various chain groups.