The Cup Product: The one big difference between homology and cohomology is that cohomology can be endowed with a "natural" product, making cohomology, specifically $\oplus_n H^n(X;R)$ into a ring. (Any group can be given "unnatural" products, like the product of any two elements are 0.)

In order to multiply cochains we will need to multiply their coefficients, and so do to this right we need to use a ring for our coefficients, instead of just an abelian group. On the other hand, most of our coefficient groups have been the additve groups of rings, anyway. Popular choices are $\mathbb{Z}, \mathbb{Z}_n$ for various n ($n = 2$ is popular), and $\mathbb{Q}$.

The basic idea is that cochains $\varphi \in \mathrm{Hom}(C_k(X), R), \psi \in \mathrm{Hom}(C_\ell(X), R)$ can be used to build a $(k+\ell)$-cochain $\varphi \cup \psi$, defined on a singular $(k+\ell)$-simplex $\sigma : \Delta^{k+\ell} = [v_0, \dots, v_{k+\ell}] \to X$ by

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,v_{k+\ell}]})$$

where the multiplication on the right takes place in R. [Technically, these restricted maps have the wrong domains; they aren't the <u>standard</u> k- and ℓ-simplices. But we just pre-compose with the "obvious" maps from the standard simplices.] This *cup product* will induce a product on cohomology, by the following fact:

... wait for it ...

$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi$.

This is essentially a routine computation. For $\sigma : \Delta^{k+\ell+1} = [v_0, \ldots, v_{k+\ell+1}] \to X$,

$$
\begin{aligned}
(\delta\varphi \smile \psi)(\sigma) &= \delta\varphi(\sigma|_{[v_0,\ldots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\ldots,v_{k+\ell+1}]}) \\
&= \varphi(\partial\sigma|_{[v_0,\ldots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\ldots,v_{k+\ell+1}]}) \\
&= \varphi(\textstyle\sum_{i=0}^{k+1}(-1)^i\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\ldots,v_{k+\ell+1}]}) \\
&= \textstyle\sum_{i=0}^{k+1}(-1)^i\varphi(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\ldots,v_{k+\ell+1}]}) = (*), \text{ while}
\end{aligned}
$$

$$
\begin{aligned}
(-1)^k(\varphi \smile \delta\psi)(\sigma) &= (-1)^k\varphi(\sigma|_{[v_0,\ldots,v_k]})\delta\psi(\sigma|_{[v_k,\ldots,v_{k+\ell+1}]}) \\
&= (-1)^k\varphi(\sigma|_{[v_0,\ldots,v_k]})\psi(\partial\sigma|_{[v_k,\ldots,v_{k+\ell+1}]}) \\
&= (-1)^k\varphi(\sigma|_{[v_0,\ldots,v_k]})\psi(\textstyle\sum_{i=k}^{k+\ell+1}(-1)^{i-k}\sigma|_{[v_k,\ldots,\hat{v_i},\ldots,v_{k+\ell+1}]}) \\
&= \textstyle\sum_{i=k}^{k+\ell+1}(-1)^i\varphi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[v_k,\ldots,\hat{v_i},\ldots,v_{k+\ell+1}]}) = (**). \text{ But then}
\end{aligned}
$$

$$
\begin{aligned}
\delta(\varphi \smile \psi)(\sigma) &= \varphi \smile \psi(\partial\sigma) = (\varphi \smile \psi)(\textstyle\sum_{i=0}^{k+\ell+1}(-1)^i\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+\ell+1}]}) \\
&= \textstyle\sum_{i=0}^{k+\ell+1}(-1)^i\varphi \smile \psi(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+\ell+1}]}) \\
&= \textstyle\sum_{i=0}^{k}(-1)^i\varphi \smile \psi(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+\ell+1}]}) + \sum_{i=k+1}^{k+\ell+1}(-1)^i\varphi \smile \psi(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+\ell+1}]}) \\
&= \textstyle\sum_{i=0}^{k}(-1)^i\varphi(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\ldots,v_{k+\ell+1}]}) \\
&\qquad\qquad + \textstyle\sum_{i=k+1}^{k+\ell+1}(-1)^i\varphi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[v_k,\ldots,\hat{v_i},\ldots,v_{k+\ell+1}]}) \\
&\overset{!}{=} \textstyle\sum_{i=0}^{k+1}(-1)^i\varphi(\sigma|_{[v_0,\ldots,\hat{v_i},\ldots,v_{k+1}]})\psi(\sigma|_{[v_{k+1},\ldots,v_{k+\ell+1}]}) \\
&\qquad\qquad + \textstyle\sum_{i=k}^{k+\ell+1}(-1)^i\varphi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[v_k,\ldots,\hat{v_i},\ldots,v_{k+\ell+1}]})
\end{aligned}
$$

$= (*) + (**)$, since

$$
\begin{aligned}
&(-1)^{k+1}\varphi(\sigma|_{[v_0,\ldots,\ldots,v_k,\widehat{v_{k+1}}]})\psi(\sigma|_{[v_{k+1},\ldots,v_{k+\ell+1}]}) \\
&\qquad\qquad + (-1)^k\varphi(\sigma|_{[v_0,\ldots,v_k]})\psi(\sigma|_{[\hat{v_k},v_{k+1},\ldots,v_{k+\ell+1}]}) = 0.
\end{aligned}
$$

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi$$

This tells us several things. first, if φ and ψ are both cocycles, then $\delta\varphi = 0$ and $\delta\psi = 0$, so $\delta(\varphi \smile \psi) = 0 \smile \psi \pm \varphi \pm 0 = 0 \pm 0 = 0$, so $\varphi \smile \psi$ is also a cocycle. therefore, the map

$$\cup : C^k(X;R) \times C^\ell(X;R) \to C^{k+\ell}(X;R)$$

induces a map

$$Z^k \times Z^\ell \to Z^{k+\ell} \to H^{k+\ell}(X;R)$$

and if either $\varphi = \delta f$ or $\psi = \delta g$, then, e.g., $\delta(f \smile \psi) = (\delta f) \smile \psi \pm f \smile \delta\psi = \varphi \smile \psi + f \smile 0 = \varphi \smile \psi$ (assuming that $\psi \in Z^\ell$), and similarly $\delta((-1)^k \varphi \smile g) = \varphi \smile \psi$, so $B^k \times Z^\ell \cup Z^k \times B^\ell$ maps into $B^{k+\ell}$, and so there is an induced map

$$\smile : H^k(X;R) \times H^\ell(X;R) \to H^{k+\ell}(X;R)$$

which is what we call the *cup product* on cohomology.

This product turns $H^*(X;R) = \oplus_n H^n(X;R)$ into (what we will call a *graded*) ring; it is associative and distributive since it is on the level of cochains;

$$(\varphi \smile \psi) \smile \theta(\sigma) = (\varphi(\sigma|_{[v_0,\dots,v_k]})\psi(\sigma|_{[v_k,\dots,v_{k+\ell}]}))\theta(\sigma|_{[v_{k+\ell},\dots,v_{k+\ell+m}]})$$
$$= \varphi(\sigma|_{[v_0,\dots,v_k]})(\psi(\sigma|_{[v_k,\dots,v_{k+\ell}]})\theta(\sigma|_{[v_{k+\ell},\dots,v_{k+\ell+m}]})) = (\varphi \smile \psi) \smile \theta(\sigma)$$

and distributivity is similar. If the coefficient ring R has an identity 1, then so does $H^*(X;R)$; the class $[1] \in H^0(X;R)$ which sends each singular 0-simplex to 1 is the identity element.

If we were to work with the simplicial cochain complex, we could define the exact same product, and so the restriction of singular cochains to simplical ones can be viewed as a ring homomorphism, and so the isomorphism between singular and simplicial cohomology is in fact a ring isomorphism. The product structure is also "natural" with respect to the maps induced by continuous maps, so $f : X \to Y$ induces a ring homomorphism $f^* : H^*(Y; R) \to H^*(X; R)$. Taking this to its logical conclusion, any homotopy equivalence induces a ring isomorphism between the respective cohomology rings.

The cup product is not quite commutative; the precise statement is that, if R is commutative, for $\varphi \in H^k(X; R)$ and $\psi \in H^\ell(X; R)$, then

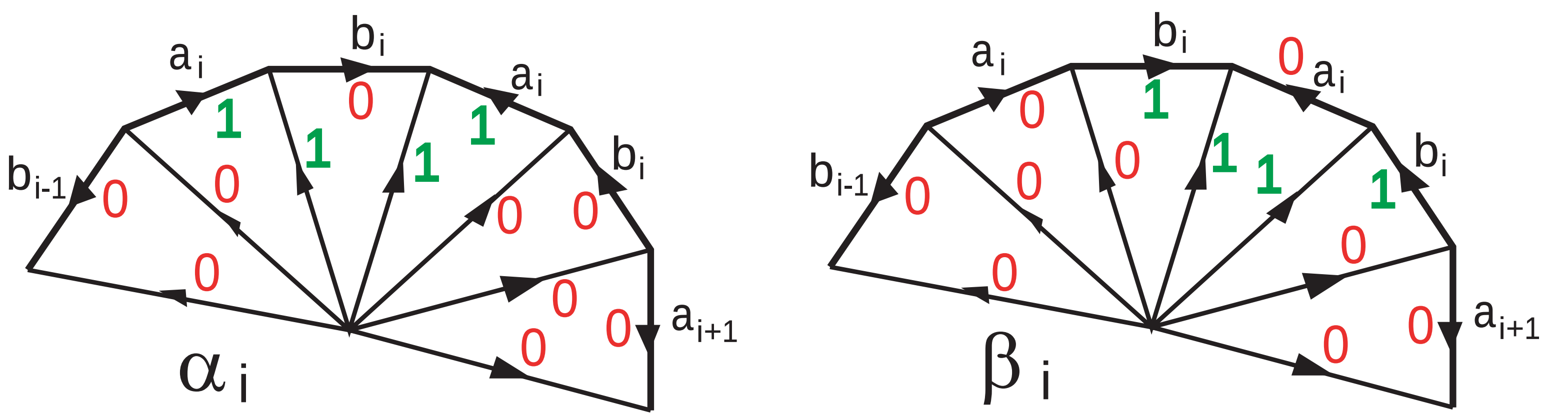
$$\varphi \smile \psi = (-1)^{k\ell} \psi \smile \varphi \in H^{k+\ell}(X; R).$$

We omit the proof. Such a product is, in some circles, called *graded commutative.*

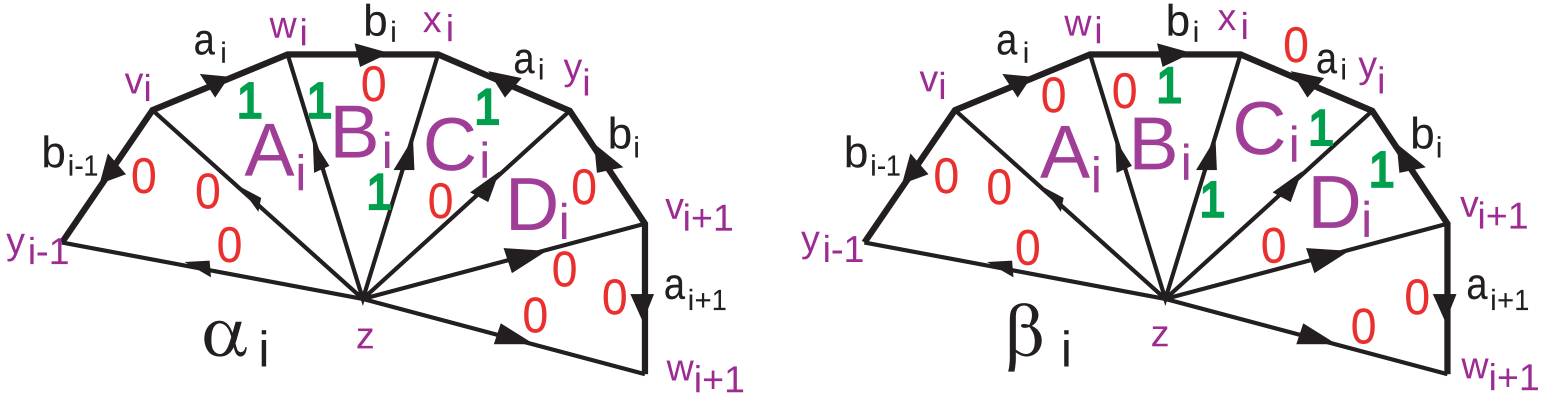
As a sample computation of the cup product for a space, we look at the closed orientable surfaces of genus $g \geq 1$, F_g. By universal coefficients, since $H_*(F_g; \mathbb{Z})$ is free abelian, all Ext groups will be 0, so we have $H^*(F_g; R) \cong R$ in dimensions 0 and 2, and $\cong R^{2g}$ in dimension 1; all other groups are 0. So the only non-trivial cup products will occur between 1-dimensional classes. Thinking in terms of cellular cohomology, using the standard CW-structure on F_g as the quotient of a $2g$-gon D with edges identified in the pattern $a_1, b_1, \overline{a_1}, \overline{b_1}, a_2, b_2, \ldots$. Identifying $H^1(F_g; R) \cong \text{Hom}(H_1(F_g), R)$ with R^{2g} as an assignment of elements of R to each of the standard basis elements a_i, b_i of $H_1(F_g)$, the cup product $\varphi \smile \psi$ can be identified with the value it assigns to the generator $[D/\partial D]$ of $H^2(F_g; R)$. It suffices to compute the cup products among the basis elements $\alpha_i = a_i^*, \beta_j = b_j^*$ dual to our basis for $H_1(F_g; R)$.

We didn't actually <u>define</u> cup products for cellular cohomology (except through its isomorphism with singular and simplicial cohomology), but we can see by the isomorphisms that simplicially, writing F_g as a Δ-complex by cutting D into $2g$ 2-simplices by coning each edge to a vertex in the center of D, we have the same basis for $H_1(F_g;R)$ and hence for $H^1(F_g;R)$, and the generator for $H_2(F_g;R)$ is the (oriented) sum of the $2g$ 2-simplices formed (since these add up to D). So to compute cup products, it suffices to determine what value of each of $\alpha_i \cup \alpha_j, \alpha_i \cup \beta_j, \beta_i \cup \alpha_j, \beta_i \cup \beta_j$ take on these sums of simplices.

Note that on the level of cochains, we must also assign the α_i, β_j values on the 1-simplices added in the interior of D. In order to be sure we are describing a cocycle, the resulting values must sum to zero around every one of the 2-simplices. The figures give one set of choices:



With these in hand, the rest is just a bunch of calculations.



$\alpha_i \smile \beta_i(A_i) = \alpha_i \smile \beta_i([z, w_i, v_i]) = \alpha_i[z, w_i]\beta_i[w_i, v_i] = 1 \cdot 0 = 0$,
$\alpha_i \smile \beta_i(B_i) = \alpha_i \smile \beta_i([z, x_i, w_i]) = \alpha_i[z, x_i]\beta_i[x_i, w_i] = 1 \cdot -1 = -1$,
$\alpha_i \smile \beta_i(C_i) = \alpha_i \smile \beta_i([z, y_i, x_i]) = \alpha_i[z, y_i]\beta_i[y_i, x_i] = 0 \cdot 0 = 0$, and
$\alpha_i \smile \beta_i(D_i) = \alpha_i \smile \beta_i([z, v_{i+1}, y_i]) = \alpha_i[z, v_{i+1}]\beta_i[v_{i+1}, y_i] = 0 \cdot 1 = 0$.
All other 2-simplices have the value 0 since all of their edges are labeled 0 by both α_i and β_i. So, summing, $\alpha_i \smile \beta_i[D] = -1$.
[$\beta_i \smile \alpha_i = 1$ follows by another computation or graded commutativity.]
Similar computations establish that

$$\alpha_i \smile \alpha_j = \alpha_i \smile \beta_j = \beta_i \smile \alpha_j = \beta_i \smile \beta_j = \alpha_i \smile \alpha_i = \beta_i \smile \beta_i = 0 \text{ for all } i, j.$$

This shows, for example, that F_2 and $S^2 \vee S^1 \vee S^1 \vee S^1 \vee S^1$ are not homotopy equivalent, even though they have isomorphic homology and cohomology groups (for all coefficients!). This is because the ring structure of
$H^*(S^2 \vee S^1 \vee S^1 \vee S^1 \vee S^1; R)$ is different; all of the cup products of 1-dim'l classes are 0. [Think of the 2-sphere as ∂(3-simplex), and note that the duals of the homology classes from the S^1s can be given values 0 on the 1-simplices of the 2-sphere. Since cup products are computed around the boundaries of the 2-simplices, they are all 0.]

The Cap Product: There is also a product which mixes cohomology and homology, and is defined in a similar way. The *cap product* of a singular chain $\sigma \in C_n(X;R)$ and a singular cochain $\varphi \in C^k(X;R)$ produces a singular chain $\sigma \frown \varphi \in C_{n-k}(X;R)$ defined by, letting $\sigma : [v_0, \dots, v_n] \to R$,

$$\sigma \frown \varphi = \varphi(\sigma|_{[v_0,\dots,v_k]})\sigma|_{[v_k,\dots,v_{k+n}]}$$

where $[v_k, \dots, v_{k+n}]$ is identified with the "standard" $(n-k)$-simplex in the obvious way. We extend this definition to n-chains R-linearly. A very similar computation to the one just carried out establishes that

$$\partial(\sigma \frown \varphi) = (-1)^n(\partial\sigma \frown \varphi - \sigma \frown \delta\varphi)$$

As before, this implies that the cap product of a cycle and a cocycle is a cycle, and if either z is a boundary or φ is a coboundary then $z \frown \varphi$ is a boundary, so we get an induced map

$$\frown : H_n(X;R) \times H^k(X;R) \to H_{n-k}(X;R)$$

which is R-linear in each coordinate, which we call the *cap product*.

The cap product is also natural with respect to continuous maps, although in an odd way: given $f : X \to Y$ we have homomorphisms

$$f_* : H_*(X;R) \to H_*(Y;R) \text{ and } f^* : H^*(Y;R) \to H^*(X;R), \text{ and}$$

$$f_*([z] \cap f^*[\varphi]) = f_*[z] \cap [\varphi]$$

The proof is similar to our argument for cup products.

The two products, cup and cap, have relative versions, which we will not explore. There is also a very concise expression relating the two products together:

If $z \in H_n(X;R), \varphi \in H^k(X;R)$, and $\psi \in H^\ell(X;R)$, then

$$z \frown (\varphi \smile \psi) = (z \frown \varphi) \frown \psi \in H_{n-k-\ell}(X;R) \ .$$

The proof is immediate; the formula holds on the level of chains and cochains.