Orientations: The cap product plays an especially important role in connecting the homology and cohomology groups of *orientable manifolds*. An *n-manifold M* is a 2^{nd} countable, Hausdorff space with the additional property that every point $x \in M$ has an open neighborhood $\mathcal{U}$ homeomorphic to $\mathbb{R}^n$. [Note: by this definition, D^n is not a manifold! It is instead a "manifold with boundary", which have their own parallel and very similar theory.] Using excision we find that for any n-manifold M and $x \in M$,

$$\begin{aligned} H_k(M, M \setminus x; R) &\cong H_k(\mathcal{U}, \mathcal{U} \setminus x; R) \cong H_k(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \\ &\cong H_k(D^n, D^n \setminus 0; R) \cong H_k(D^n, \partial D^n; R) \cong \widetilde{H}_k(D^n/\partial D^n; R) \cong \widetilde{H}_k(S^n; R) \end{aligned}$$

and so equals 0 for $k \neq n$ and R for $k = n$. An *R-orientation* on an n-manifold M is a choice of generator (as an R-module) $r_x \in H_n(M, M \setminus x; R) \cong R$ (called a *local R-orientation* at x) for every $x \in M$, which are locally compatible: for every $x \in M$ there is a nbhd $\mathcal{U}$ of x and $r_{\mathcal{U}} \in H_n(M, M \setminus \mathcal{U}; R)$ such that, for every $y \in \mathcal{U}$, the inclusion-induced map $\iota_* : H_n(M, M \setminus \mathcal{U}; R) \to H_n(M, M \setminus y; R)$ sends $r_{\mathcal{U}}$ to r_y.

For example, every manifold is $\mathbb{Z}_2$-orientable; a local $\mathbb{Z}_2$-orientation is a choice of the (one and only) non-zero element of $H_n(M, M \setminus x; \mathbb{Z}_2)$, and for any pair of locally Euclidean nbhds $\mathcal{U} \subseteq \overline{\mathcal{U}} \subseteq \mathcal{V}$ of x, and $y \in \mathcal{U}$,

$$H_n(M, M \setminus \mathcal{U}; \mathbb{Z}_2) \cong H_n(\mathcal{V}, \mathcal{V} \setminus \mathcal{U}; \mathbb{Z}_2) \cong H_n(\mathcal{V}, \mathcal{V} \setminus y; \mathbb{Z}_2) \cong H_n(M, M \setminus y; \mathbb{Z}_2) \cong \mathbb{Z}_2$$

where all isomorphisms are inclusion-induced, and the second assumes that $\mathcal{U}$ is a ball in $\mathcal{V}$, so that $\overline{\mathcal{U}} \setminus y$ deformation retracts to $\partial\overline{\mathcal{U}} \subseteq \mathcal{V}$, so $\mathcal{V} \setminus y$ deformation retracts to $\mathcal{V} \setminus \mathcal{U}$. Consequently, the inclusion-induced isomorphism $H_n(M, M \setminus \mathcal{U}; \mathbb{Z}_2) \cong H_n(M, M \setminus y; \mathbb{Z}_2)$ sends the non-zero element of $H_n(M, M \setminus \mathcal{U}; \mathbb{Z}_2)$, which we define to be $r_{\mathcal{U}}$, to the non-zero element of $H_n(M, M \setminus y; \mathbb{Z}_2)$, which is r_y.

As an example, the closed orientable surfaces F_g of genus g are R-orientable for every R (hence the name...); from the LES for a pair we have

$$\cdots \to H_2(F_g \setminus x; R) \to H_2(F_g; R) \to H_2(F_g, F_g \setminus x; R) \to H_1(F_g \setminus x; R) \to H_1(F_g; R) \to \cdots$$

which, via universal coefficients and because $F_g \setminus x$ deformation retracts to the 1-skeleton, is

$$0 \to R \xrightarrow{i_*} R \to R^{2g} \xrightarrow{j_*} R^{2g} \ .$$

But j_* is an isomorphism (the 2-cell has boundary 0, so there "are" no 1-boundaries), therefore by exactness so is i_*, so the image of a generator of $H_2(F_g)$ defines a (compatible: the open cover <u>is</u> F_g) set of local orientations at each point.

The first basic fact about orientations is that what just happened is not an accident; if M^n is R-orientable and *compact*, then there is a (unique!) homology class $[M] \in H_n(M; R) = H_n(M, \emptyset; R)$, the *orientation class* of M, such that the image of $[M]$ in $H_n(M, M \setminus x; R)$ defines the same orientation on M (i.e., it equals r_x for every x).

To prove this, start with ball neighborhoods $\mathcal{U}_x$ of each point as an open cover, and take smaller ball neighborhoods $\mathcal{V}_x \subseteq \overline{\mathcal{V}_x} \subseteq \mathcal{U}_x$, with compact closures (e.g., the inverse image of the unit ball under a homeo $h : \mathcal{U}_x \to \mathbb{R}^n$) as another open cover. By compactness, finitely many $\{\mathcal{V}_i\}_{i=1}^m$ of the $\mathcal{V}_x$ cover M. For notational sanity, let us write $H_n(M|A)$ for $H_n(M, M \setminus A; R)$. The isomorphisms $H_n(M|\mathcal{U}_i) \cong H_n(M|\overline{\mathcal{V}_i}) \cong H_n(M, |y) \cong R$ (since each subspace deformation retracts to the next smaller one) implies that there is a unique class $r_i \in H_n(M|\overline{\mathcal{V}_i})$ (the image of the class $r_{\mathcal{U}_i} \in H_n(M|\mathcal{U}_i)$) which maps to each r_y under inclusion (unique because we have isos).

For notational convenience, we set $K_i = \overline{\mathcal{V}_i}$. We wish to show that there is a unique class $[M]$ in $H_n(M;R) = H_n(M|M)$ which restricts to each of the r_i; further restriction then implies that it maps to each r_x, as desired. First we prove uniqueness. Suppose there were two classes u, v restricting to each of the r_i. Then their difference, $u - v = w$, restricts to 0 in every $H_n(M|K_i)$. We show by induction that then w restricts to 0 in the groups $G_j = H_n(M|K_1 \cup \cdots \cup K_j)$. But since $G_m = H_n(M|\bigcup_i K_i) = H_n(M|M) = H_n(M;R)$, $w = 0$ as desired. For the inductive step, we use the relative Mayer-Vietoris sequence

$$\cdots \to H_{n+1}(M|(K_1 \cup \cdots \cup K_i) \cap K_{i+1}) \to H_n(M|(K_1 \cup \cdots \cup K_i) \cup K_{i+1})) \to \\ H_n(M|K_1 \cup \cdots \cup K_i) \oplus H_n(M|K_{i+1}) \to H_n(M|(K_1 \cup \cdots \cup K_i) \cap K_{i+1}) \to \cdots$$

A separate induction (which we skip) establishes that $H_{n+1}(M|K_1 \cup \cdots \cup K_i) \cap K_{i+1}) = H_{n+1}(M|(K_1 \cap K_{i+1}) \cup \cdots \cup (K_i \cap K_{i+1})) = 0$. So $H_n(M|(K_1 \cup \cdots \cup K_i) \cup K_{i+1}))$ injecting into $H_n(M|K_1 \cup \cdots \cup K_i) \oplus H_n(M|K_{i+1})$. But the image of w in $H_n(M|(K_1 \cup \cdots \cup K_i) \cup K_{i+1}))$ is then carried to 0 in both $H_n(M|K_1 \cup \cdots \cup K_i)$ and $H_n(M|K_{i+1})$ by the inductive hypothesis (and initial step), so by injectivity is itself 0, establishing the inductive step.

From uniqueness, we can go on to establish existence, again by induction. Note that the uniqueness argument above applies more generally; for any compact set $K \subseteq M$ there is at most one class $r_K \in H_n(M|K)$ which restricts to $r_x \in H_n(M|x)$ for every $x \in K$. This essentially allows us to stitch together the classes which compatibility guarantees exist for small K to ever larger K. Formally, we just parallel the argument above; given $M = K_1 \cup \cdots \cup K_m$, we have classes $r_i \in H_n(M|K_i)$ which restrict to the local orientations. The point is that in the relative Mayer-Vietoris sequence

$$H_n(M|K_{m-1} \cup K_m) \to H_n(M|K_{m-1}) \oplus H_n(M|K_m) \to H_n(M|K_{m-1} \cap K_m)$$

the classes r_{m-1}, r_m each map to a class in $H_n(M|K_{m-1} \cap K_m)$ (under the inclusion-induced maps) which restricts to r_x for every $x \in K_{m-1} \cap K_m$, and so by uniqueness map to the same class. So in the Mayer-Vietoris sequence, (r_{m-1}, r_m) maps to 0, and so is in the image of $H_n(M|K_{m-1} \cup K_m)$, so there is a class $r'_{m-1} \in H_n(M|K_{m-1} \cup K_m)$ which restricts to $r - x$ for every $x \in K_{m-1} \cup K_m$. Now replace K_{m-1}, K_m by $K_{m-1} \cup K_m$ in our cover of M by compact sets, and continue (or declare victory!) by induction, since we have a cover by fewer sets having the hypothesized classes r_i; once we reach one such set, we have $K_1 = M$.

Given a compact, connected R-orientable n-manifold, we now have an orientation class $[M] \in H_n(M;R)$ which defines an orientation on M. This class plays a central role in

Poincaré Duality: If M is a compact, connected, R-orientable n-manifold, then for every k, the map $P[\varphi] = [M] \frown [\varphi]$, $P : H^k(M;R) \to H_{n-k}(M;R)$ is an isomorphism.

We will not prove this; the proof is in many respects parallel to the one given above, inducting on a number of "small" compact subsets whose union is M, but it formally requires introducing a new cohomology theory, *cohomology with compact supports*, which we will not take the time to explore. Instead, we will outline some of the consequences of this result.

For a connected R-orientable compact n-manifold M, $H_n(M;R) \cong H^0(M;R) \cong \mathrm{Hom}(H_0(M),R) \oplus \mathrm{Ext}(H_{-1}(M),R) \cong \mathrm{Hom}(\mathbb{Z},R) \oplus \mathrm{Ext}(0,R) \cong R$. [Note that this immediately implies that $\mathbb{R}P^{2k}$ is not orientable, since $H_{2k}(\mathbb{R}P^{2k} = 0$.] This is really a kind of cheap consequence, though, because in proving Poincaré duality, you basically have to prove this first...

For a connected $\mathbb{Z}$-orientable compact manifold M, $H_{n-1}(M;\mathbb{Z}) \cong H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(H_1(M),\mathbb{Z}) \oplus \mathrm{Ext}(H_0(M),\mathbb{Z}) \cong \mathrm{Hom}(H_1(M),\mathbb{Z}) \oplus \mathrm{Ext}(\mathbb{Z},\mathbb{Z}) = \mathrm{Hom}(H_1(M),\mathbb{Z})$ $=$ the torsion-free part of $H_1(M)$, so $H_{n-1}M)$ is, in particular, torsion-free. [Note that this also immediately implies that $\mathbb{R}P^{2k}$ is not orientable, since $H_{2k-1}(\mathbb{R}P^{2k} \cong \mathbb{Z}_2$.]

We've seen that the Euler characteristic can be computed using a singular chain complex with coefficients in any (non-trivial!) field, for finite CW-complexes. Using the (somewhat hard) fact that a compact manifold admits the structure of a finite CW-complex, we can then use this to learn a few things about the Euler characteristics of compact manifolds. Since Poincare duality holds for any compact manifold M when using $\mathbb{Z}_2$ coefficients, we have

$$H_{n-k}(M;\mathbb{Z}_2) \cong H^k(M;\mathbb{Z}_2) \cong \mathrm{Hom}(H_k(M),\mathbb{Z}_2) \oplus \mathrm{Ext}(H_{k-1}(M),\mathbb{Z}_2)$$

But we have seen that $H_k(M;\mathbb{Z}_2) \cong H_k(M)/\{2[z] : [z] \in H_k(M)\} \oplus \{[z] \in H_{k-1}(M) : 2[z] = 0\}$, and a quick bit of algebra will show that, at least for finitely generated abelian groups G, $G/\{2z : z \in G\} \cong \mathrm{Hom}(G,\mathbb{Z}_2)$ and $\{z \in G : 2z = 0\} \cong \mathrm{Ext}(G,\mathbb{Z}_2)$ (since everybody behaves well under direct sums and these isomorphisms hold for $\mathbb{Z}$ and $\mathbb{Z}_n$). So $H_{n-k}(M;\mathbb{Z}_2) \cong H_k(M;\mathbb{Z}_2)$. Consequently, when computing Euler characteristic, for an odd-dimensional manifold, we find that (using dimension as a $\mathbb{Z}_2$-vector space)

$$\begin{aligned}
\chi(M) &= \textstyle\sum(-1)^i \dim(H_i(M;\mathbb{Z}_2)) \\
&= \textstyle\sum_{i<n/2}(-1)^i \dim(H_i(M;\mathbb{Z}_2)) + \sum_{i>n/2}(-1)^i \dim(H_i(M;\mathbb{Z}_2)) \\
&= \textstyle\sum_{i<n/2}(-1)^i \dim(H_i(M;\mathbb{Z}_2)) + \sum_{i>n/2}(-1)^{-i} \dim(H_{n-i}(M;\mathbb{Z}_2)) \\
&= \textstyle\sum_{i<n/2}(-1)^i \dim(H_i(M;\mathbb{Z}_2)) - \sum_{i>n/2}(-1)^{n-i} \dim(H_{n-i}(M;\mathbb{Z}_2)) \\
&= \textstyle\sum_{i<n/2}(-1)^i \dim(H_i(M;\mathbb{Z}_2)) - \sum_{j<n/2}(-1)^j \dim(H_j(M;\mathbb{Z}_2)) \\
&= 0\ .
\end{aligned}$$

So odd-dimensional compact manifolds (without boundary) M all have Euler characteristic 0. $\chi(M) = 0$ then holds no matter how we (correctly) compute it, so this tells us something about the ranks of the $\mathbb{Z}$-homology groups, as well as about the CW-structure itself on M.