Universal covering spaces: A particularly important covering space of a space X to identify is one which is simply connected. Such a covering is essentially unique:

If X is *locally path-connected* and has two connected, simply connected covering spaces $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$, then since $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(\pi_1(X_2, x_2)) = \{1\} \subseteq \pi_1(X, x_0)$, the lifting criterion applied twice gives maps $\widetilde{p}_1 : (X_1, x_1) \to (X_2, x_2)$ and $\widetilde{p}_2 : (X_2, x_2) \to (X_1, x_1)$ with $p_2 \circ \widetilde{p}_1 = p_1$ and $p_1 \circ \widetilde{p}_2 = p_2$. Consequently, $p_2 \circ \widetilde{p}_1 \circ \widetilde{p}_2 = p_1 \circ \widetilde{p}_2 = p_2$ and similarly, $p_1 \circ \widetilde{p}_2 \circ \widetilde{p}_1 = p_2 \circ \widetilde{p}_1 = p_1$. So $\widetilde{p}_1 \circ \widetilde{p}_2 : (X_2, x_2) \to (X_2, x_2)$, for example, is a lift of p_2 to the covering map p_2. But so is the identity map! By uniqueness, therefore, $\widetilde{p}_1 \circ \widetilde{p}_2 = Id$. Similarly, $\widetilde{p}_2 \circ \widetilde{p}_1 = Id$. So (X_1, x_1) and (X_2, x_2) are homeomorphic. (More: there is a homeo interpolating between the covering maps.) So up to homeomorphism, a space has only one connected, simply-connected covering space. It is known as the *universal covering* of the space X.

Not every (locall path-connected) space X has a universal covering; a (further) necessary condition is that X be *semi-locally simply connected*. The idea is that If $p : \widetilde{X} \to X$ is the universal cover, then for every point $x \in X$, we have an evenly-covered neighborhood $\mathcal{U}$ of x. The inclusion $i : \mathcal{U} \to X$, by definition, lifts to $\widetilde{X}$, so $i_*(\pi_1(\mathcal{U}, x)) \subseteq p_*(\pi_1(\widetilde{X}, \widetilde{x}) = \{1\}$, so i_* is the trivial map. Consequently, every loop in $\mathcal{U}$ is null-homotopic in X. This is semi-local simple connectivity; every point has a neighborhood whose inclusion-induced homomorphism is trivial. Not all spaces have this property; the most famous is the Hawaiian earrings $X = \bigcup_n \{x \in \mathbb{R}^2 : ||x - (1/n, 0)|| = 1/n\}$. The point $(0, 0)$ has no such neighborhood.

Building universal coverings: If a space X is path connected, locally path connected, and semi-locally simply connected (S-LSC), then it has a universal covering.

The idea is that a covering space has the path lifting and homotopy lifting properties, and the universal cover is the only covering space for which *only* null-homotopic loops lift to loops. So we build a space and a map which must have these properties. We do this by making a space $\widetilde{X}$ whose points are (equivalence classes $[\gamma]$ of) based paths $\gamma : (I, 0) \to (X, x_0)$, where two paths are equivalent if they are homotopic rel endpoints! The projection map is $p([\gamma]) = \gamma(1)$.

The S-LSCness of X guarantees that this is a covering map; choosing $x \in X$, a path γ_0 from x_0 to x, and a nbhd $\mathcal{U}$ of x guaranteed by S-LSC, a path θ from x_0 to points in $\mathcal{U}$ is homotopic to $\gamma * \gamma_0 * \eta$ where γ is a loop at x_0 and η is a path in $\mathcal{U}$. $[\gamma = \theta * \overline{\eta} * \overline{\gamma_0}.]$ But by S-LSC, a path in $\mathcal{U}$ is determined up to homotopy by its endpoints, and so, with γ fixed, these paths are in one-to-one correspondence with $\mathcal{U}$. So $p^{-1}(\mathcal{U})$ is a disjoint union, indexed by $\pi_1(X, x_0)$, of sets that are in 1-to-1 corresp with $\mathcal{U}$.

The appropriate topology on $\widetilde{X}$ is essentially given as a basis by triples $(\gamma, \gamma_0, \mathcal{U})$ as above. This topology makes p a covering map. Note that the inverse image of the basepoint x_0 is the equivalence classes of loops at x_0, i.e., $\pi_1(X, x_0)$. A path γ lifts to the path of paths γ_t, where $\gamma_t(s) = \gamma(ts)$, and so the only loops in X which lift to a loop in $\widetilde{X}$ have $[\gamma_1] = [\gamma] = [c_{x_0}]$, i.e., $[\gamma] = 1$ in $\pi_1(X, x_0)$. This implies that $p_*(\pi_1(\widetilde{X}, [c_{x_0}])) = \{1\}$, so $\pi_1(\widetilde{X}, [c_{x_0}]) = \{1\}$.

However, nobody in their right minds would go about building $\widetilde{X}$ in this way!

Why Care? The universal cover gives a unified approach to building <u>all</u> connected covering spaces of X. The key to this is the *deck transformation group (Deckbewegungsgruppe)* of a covering space $p : \widetilde{X} \to X$; this is the set of all homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $p \circ h = p$.

By def'n, these h permute each of the pt inverses of p. Since h is a lift of the projection map p, by the lifting criterion h is det'd by which point in $p^{-1}(x_0)$ it takes the basepoint $\widetilde{x}_0$ of $\widetilde{X}$ to. A deck transformation sending $\widetilde{x}_0$ to $\widetilde{x}_1$ exists $\Leftrightarrow p_*(\pi_1(\widetilde{X}, \widetilde{x}_0) = p_*(\pi_1(\widetilde{X}, \widetilde{x}_1)$ [we need one inclusion to give h, and the opposite inclusion to ensure it is a bijection].

These two groups are *conjugate*, by the projection of a path from $\widetilde{x}_0$ to $\widetilde{x}_1$ (follow the change of basept iso down into $G = \pi_1(X, x_0)$). Paths in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$ are in 1-to-1 corresp with the cosets of $H = p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)$ in $p_*(\pi_1(X, x_0)$; so deck transformations are in 1-to-1 corresp with cosets whose representatives conjugate H to itself. The set of such elements in G is called the *normalizer of H in G*, and denoted $N_G(H)$ or simply $N(H)$. The deck transformation group is therefore in 1-to-1 correspondence with the group $N(H)/H$ under $h \mapsto$ the coset with representative the projection of the path from $\widetilde{x}_0$ to $h(\widetilde{x}_0)$. And since the lift h is essentially built by lifting paths, it follows quickly that this map is a homomorphism, hence an isomorphism.

Applying this to the universal covering space $p : \widetilde{X} \to X$, in this case $H = \{1\}$, so $N(H) = \pi_1(X, x_0)$. So the deck transformation group is isomorphic to $\pi_1(X, x_0)$. For example, this gives the quickest possible proof that $\pi_1(S^1) \cong \mathbb{Z}$, since $\mathbb{R}$ is a contractible covering space, whose deck transformations are the translations by integer distances.

Thus $\pi_1(X)$ acts on its universal cover as a group of homeomorphisms. And since this action is *simply transitive* on point inverses [there is exactly one (that's the simple part) deck transformation carrying any one point in a point inverse to any other one (that's the transitive part)], the quotient map from $\widetilde{X}$ to the orbits of this action <u>is</u> the projection map p. The evenly covered property of p implies that X does have the quotient topology under this action.

So every space it X the quotient of its universal cover (if it has one!) by its fundamental group $G = \pi_1(X, x_0)$, acting as the group of deck transformations. And the quotient map is the covering projection. So $X \cong \widetilde{X}/G$.

In general, a quotient of a space Z by a group action G need not be a covering map. The action must be *properly discontinuous*: for every point $z \in Z$, there is a neighborhood $\mathcal{U}$ of x so that $g \neq 1 \Rightarrow \mathcal{U} \cap g\mathcal{U} = \emptyset$. The evenly covered neighborhoods provide these for the universal cover. And conversely, the quotient of a space by a p.d. group action is a covering space.

But! Given $G = \pi_1(X, x_0)$ and its action on a univ cover $\widetilde{X}$, we can, instead of modding out by G, mod out by any <u>subgroup</u> H of G, to build $X_H = \widetilde{X}/H$. This is a space with $\pi_1(X_H) \cong H$, having $\widetilde{X}$ as univ covering. And since the quotient (covering) map $p_G : \widetilde{X} \to X = \widetilde{X}/G$ factors through $\widetilde{X}/H$, we have an induced map $p_H : \widetilde{X}/H \to X$, which is a covering map; open sets with trivial inclusion-induced homomorphism lift homeomorphically to $\widetilde{X}$, hence homeomorphically to $\widetilde{X}/H$; choosing lifts to each point inverse of $x \in X$ builds the evenly covering nbhds for p_H . So every subgroup of G is the fundamental group of a covering of X.

The Galois correspondence: Two coverings $p_1 : X_1 \to X$, $p_2 : X_2 \to X$ are *isomorphic* if there is a homeo $h : X_1 \to X_2$ with $p_1 = p_2 \circ h$. Choosing basepts x_1, x_2 mapping to $x_0 \in X$, then if $h(x_1) = x_2$, then $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(h_*(\pi_1(X_1, x_1))) = p_{2*}(\pi_1(X_2, x_2))$. If instead $h(x_1) = x_3$, then $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(\pi_1(X_2, x_3))$. But $\pi_1(X_2, x_2)$ and $\pi_1(X_2, x_3)$ are isomorphic, via a change of basept isomorphism $\widehat{\eta}$, where η is a path in X_2 from x_2 to x_3. Such a path projects to X as a loop at x_0, and since the change of basept isom is by "conjugating" by the path η, the resulting groups $p_{2*}(\pi_1(X_2, x_2))$ and $p_{2*}(\pi_1(X_2, x_3))$ are conjugate, by $[p_2 \circ \eta]$.

So choosing <u>any</u> basepts over x_0, isomorphic coverings give, under projection, conjugate subgroups of $\pi_1(X, x_0)$. But conversely, given covering spaces X_1, X_2 whose subgroups $p_{1*}(\pi_1(X_1, x_1))$ and $p_{2*}(\pi_1(X_2, x_2))$ are conjugate, lifting a loop γ representing the conjugating element to a loop $\widetilde{\gamma}$ in X_2 starting at x_2 gives, as its terminal endpoint, a point x_3 with $p_{1*}(\pi_1(X_1, x_1)) = p_{2*}(\pi_1(X_2, x_3))$ (since it conjugates back!), and so, by the lifting criterion, there is an isomorphism $h : (X_1, x_1) \to (X_2, x_3)$. So conjugate subgroups give isomorphic coverings. Thus:

The Galois correspondence: For a path-connected, locally path-connected, semi-locally simply-connected space X, the image of the induced homomorphism on π_1 gives a one-to-one correspondence between [isomorphism classes of (connected) coverings of X] and [conjugacy classes of subgroups of $\pi_1(X)$].

So, for example, if you have a group G that you are interested in, you know of a (nice enough) space X with $\pi_1(X) \cong G$, and you know enough about the coverings of X, then you can gain information about the subgroup structure of G.

For example, a free group $F(\Sigma)$ is π_1 of a bouquet of circles X. Any covering space $\widetilde{X}$ of X is a union of vertices and edges, so is a graph. Collapsing a maximal tree to a point, $\widetilde{X}$ is $\sim$ a bouquet of circles, so has free π_1. So every subgroup of a free group is free. A subgroup H of index n in $F(\Sigma)$ corresponds to a n-sheeted covering $\widetilde{X}$ of X. If $|\Sigma| = m$, then $\widetilde{X}$ will have n vertices and nm edges. Collapsing a maximal tree, having $n-1$ edges, to a point, leaves a bouquet of $nm-n+1$ circles, so $H \cong F(nm-n+1)$. For example, for $m=3$, index n subgroups are free on $2n+1$ generators, so every free subgroup on 4 generators has infinite index in $F(3)$. [Try proving that directly!]