Higher homotopy groups: Fundamental groups are a remarkably powerful tool for studying spaces; they capture a great deal of the global structure of a space, and so they are very good a detecting between homotopy-inequivalent spaces. In theory! **But** in practice, they suffer from the fact that deciding whether two groups are isomorphic or not is, in general, undecideable....

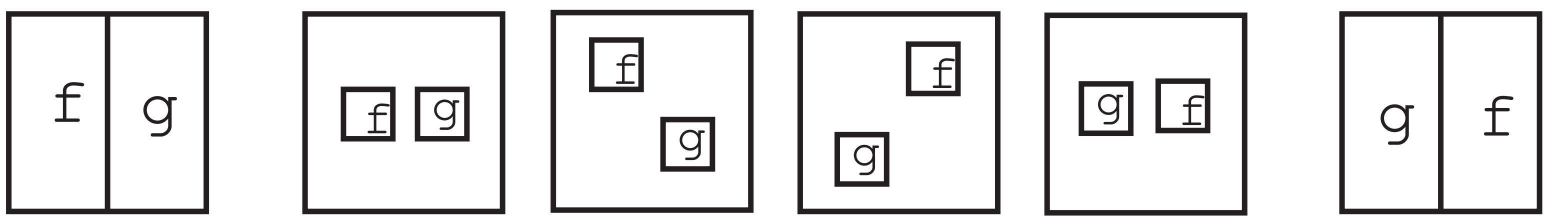
There are also "higher" homotopy groups beyond the fundamental group π_1, (hence the name pi-*one*); elements are homotopy classes, rel boundary, of based maps $\gamma : (I^n, \partial I^n) \to (X, x_0)$. Multiplication is again by concatenation. We glue two n-cubes side-by-side and then reprarametrize into a single cube:

$$\gamma * \eta(t_1, \dots, t_n) = \begin{cases} \gamma(2t_1, t_2 \dots, t_n), & \text{if } t_1 \leq 1/2 \\ \eta(2t_1 - 1, t_2 \dots, t_n) & \text{if } t_1 \geq 1/2 \end{cases}$$

The elements can be interpreted as based homotopy classes of maps $\gamma : (S^n, 1) \to (X, x_0)$, by crushing ∂I^n to a point. Like π_1, it describes, essentially, maps of S^n into X which don't extend to maps of D^{n+1}, i.e., it turns the "n-dimensional holes" of X into a group.

The (well, an) inverse reverses the first coordinate; a homotopy to the constant map is built by applying the homotopy in the π_1 case for each tuple $(t_2, \dots, t_n)$.

One feature of the higher ($n \geq 2$) homotopy groups is that they are all *abelian*: $\gamma * \eta$ is homotopic, rel boundary, to $\eta * \gamma$. The homotopy may be obtained by "spinning" the middle boundary between γ and η around one of the the coordinates.



As with π_1, a continuous map induces homomorphisms of higher homotopy groups: $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ by post-composition with f.

Another distinctive feature of the higher case is that a covering map $p : \widetilde{X} \to X$ induces an *isomorphism* on π_n; any map $\gamma : S^n \to X$ lifts, by the lifting criterion, giving surjectivity, and homotopy lifting gives injectivity as usual. As before, a homotopy equivalence induces isomorphisms of the higher homotopy groups, as well.

But unlike π_1, where we have a chance to compute it from simpler pieces via Seifert-van Kampen, nobody, for example knows what all of the homotopy groups $\pi_n(S^2)$ are (except that nearly all of them are non-trivial!). For particular spaces (except for contractible ones!) they are, as a result, notoriously difficult to compute. But this doesn't stop anyone from using them!, thanks in large part to

Whitehead's Theorem: any map between CW-complexes that induces an isomorphism on all homotopy groups is a homotopy equivalence.

Proving this result would take us too far afield, however.