Homology groups: We start by defining *n-chains*; these are (finite) formal linear combinations of the (oriented!) n-simplices of X, where $-\sigma$ is interpreted as σ with the opposite (i.e., other) orientation. Adding formal linear combinations formally, we get the n-th *chain group* $C_n(X) = \{\sum n_\alpha \sigma_\alpha : \sigma_\alpha$ an oriented n-simplex in $X\}$. We next define a *boundary operator* $\partial : C_n(X) \to C_{n-1}(X)$, whose image will be the $(n-1)$-chains that are the "boundaries" of n-chains. The idea is that the boundary of a 2-simplex, for example, should be a "sum" of its three faces (since they do make up the boundary of the simplex), "accounting for" orientations. Thinking of the orientation on a 1-simplex $[v, w]$ as an arrow pointing from v to w, we are led to believe that the boundary of a 2-simplex $[u, v, w]$ should be $[u, v] + [v, w] + [w, u]$. Similarly, the boundary of $[u, v]$, on reflection, should be $[v] - [u]$, to distinguish the head of the arrow (the $+$ side) from the tail (the $-$ side). On the basis of these examples, trying to find a consistent formula, one might eventually be led to the following definition: we define the boundary map on the basis elements $\sigma_\alpha = \sigma$ of $C_n(X)$ as $\partial\sigma = \sum(-1)^i \sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_n]}$, where $\sigma : [v_0, \dots , v_n] \to X$ is the characteristic map of σ_α . $\partial\sigma$ is therefore an alternating sum of the faces of σ. We then extend the definition by linearity to all of $C_n(X)$. When a notation indicating dimension is needed, we write $\partial = \partial_n$. We define $\partial_0 = 0$.

This definition, it turns out, is cooked up to make the maxim "boundaries have no boundary" true; that is, $\partial_{n-1} \circ \partial_n = 0$, the 0 map. This is because, for any simplex $\sigma = [v_0, \ldots v_n]$,

$$\partial \circ \partial(\sigma) = \partial(\sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]})$$

$$= (\sum_{j<i} (-1)^j (-1)^i \sigma|_{[v_0,\ldots,\widehat{v_j},\ldots,\widehat{v_i},\ldots,v_n]}) + (\sum_{j>i} (-1)^{j-1} (-1)^i \sigma|_{[v_0,\ldots,\widehat{v_i},\ldots,\widehat{v_j},\ldots,v_n]})$$

The distinction between the two pieces is that in the second part, v_j is actually the $(j-1)$-st vertex of the face. Switching the roles of i and j in the second sum, we find that the two are negatives of one another, so they sum to 0, as desired.

And this calculation is all that it takes to define homology groups. What it tells us is that $\mathrm{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$ for every n. $\ker(\partial_n = Z_n(X)$ is called the *n-cycles* of X; they are the n-chains with 0 (i.e., empty) boundary. They form a (free) abelian subgroup of $C_n(X)$. $\mathrm{im}(\partial_{n+1} = B_n(X)$ is the *n-boundaries* of X; they are, of course, the boundaries of $(n+1)$-chains in X. The n-th homology group of X, $H_n(X)$ is the quotient $Z_n(X)/B_n(X)$; it is an abelian group.

A key observation is that the boundary maps ∂_n are linear, that is, they are homomorphisms between the free abelian groups $\partial_n : C_n(X) \to C_{n-1}(X)$. Consequently, they can be expressed as (integer-valued) matrices Δ_n. Row reducing Δ_n (over the integers!) allows us to find a basis $v_1, \ldots, v_k$ for $Z_n(X)$ (clearing denomenators to get vectors over $\mathbb{Z}$). Then since $\Delta_n \Delta_{n+1} = 0$, the columns of Δ_{n+1} are in the kernel of Δ_n, so can be expressed as linear combinations of the v_i . These combinations can be determined by row reducing the augmented matrix $(v_1 \cdots v_k | \Delta_{n+1})$. This will row reduce to $\begin{pmatrix} I & | & C \\ 0 & | & 0 \end{pmatrix}$, and C basically describes the boundaries $B_n(X)$ in terms of the basis $v_1, \ldots, v_k$. The homology group $H_n(X)$ is then the *cokernel* of C, i.e., $\mathbb{Z}^k / \mathrm{im} C$. Note that C will have integer entries, since we know that the columns of Δ_{n+1} can be expressed as integer linear combinations of the v_i, and, being a basis, there is only one such expression.

Some examples: the Klein bottle K has a Δ-complex structure with 2 2-simplices, 3 1-simplices, and 1 0-simplex; we will call them $f_1 = [0,1,2], f_2 = [1,2,3], e_1 = [0,2] = [1,3], e_2 = [1,0] = [2,3], e_3 = [1,2]$, and $v_1 = [0] = [1] = [2] = [3]$. Computing, we find $\partial_2 f_1 = \partial[0,1,2] = [1,2] - [0,2] + [0,1] = e_3 - e_1 - e_2$, $\partial_2 f_2 = e_2 - e_1 + e_3$, $\partial_1 e_1 = \partial_1 e_2 = \partial_1 e_3 = 0$ and $\partial_i = 0$ for all other i (as well). So we have the chain complex
$\cdots \to 0 \to \mathbb{Z}^2 \to \mathbb{Z}^3 \to \mathbb{Z} \to 0$
and all of the boundary maps are 0, except for ∂_2, which has the matrix $\begin{pmatrix} -1 & -1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}$. This matrix is injective, so $\ker \partial_2 = 0$, so $H_2(K) = 0$, on the other hand, $H_1(K)$ $=$ coker(∂_2), and applying column operations we can transform the matrix for ∂_2 to $\begin{pmatrix} 1 & 0 \\ 1 & 2 \\ -1 & 0 \end{pmatrix}$, which implies that the cokernel is $\mathbb{Z} \oplus \mathbb{Z}_2$, since $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a basis for $\mathbb{Z}^3$. Finally, $H_0(K) = \mathbb{Z}$, since $\partial_1, \partial_0 = 0$, and all higher homology groups are also 0, for the same reason.

As another example, the topologist's dunce hat has a Δ-structure with 1 2-simplex, 1 1-simplex, and 1 0-simplex. The boundary maps, we can work out (starting from $C_2(X)$), are $(1), (0)$, and (0), so $H_2(X) = H_1(X) = 0$, and $H_0(X) = \mathbb{Z}$. all higher groups are also 0.