Homology and homotopy groups: There are connections between homology groups and the fundamental (and higher) homotopy groups, provided by what is known as the *Hurewicz map* $H : \pi_n(X, x_0) \to H_n(X)$. For $n = 1$ (higher n are similar) the idea is that elements of $\pi_1(X)$ are loops, which can be thought of as maps $\gamma : S^1 \to X$ (or more precisely, mapping into the path component containing x_0), inducing a map $\gamma_* : \mathbb{Z} = H_1(S^1) \to H_1(X)$. We define $H([\gamma]) = \gamma_*(1)$. Because homotopic maps give the same induced map on homology, this really is well-defined map on homotopy classes, i.e. from $\pi_1(X)$ to $H_1(X)$. [A different view: a loop $\gamma : (I, \partial I) \to (X, x_0)$ defines a singular 1-chain which, being a loop, has zero boundary, so is a 1-cycle. Since based homotopic maps give homologous chains (essentially by the same homotopy invariance property above), we get a well-defined map $\pi_1(X, x_0) \to H_1(X)$.
Since as 1-chains, the concatenation $\gamma * \delta$ of two loops is homologous to the sum $\gamma + \delta$ - the map $K : I \times I \to X$ given by $K(s,t) = (\gamma * \delta)(s)$, after crushing the left and right vertical boundaries to points, can be thought of as a singular 2-simplex with boundary $\gamma + \delta - (\gamma * \delta)$ - the map H is a homomorphism.

When X is path-connected, this map $H : \pi_1(X) \to H_1(X)$ is onto . [When it isn't it maps onto the summand of $H_1(X)$ corresponding to the path component containing our chosen basepoint.] To see this, note that any cycle $z \in Z_1(X)$ can be represented as a sum of singular 1-simplices $\sum \sigma_i^1$, i.e. we can (by reversing the orientations on simplices to make coefficient positive, and then writing a multiple of a simplex as a sum of simplices) assume all coefficients in our sum are 1. Then $0 = \partial z = \sum(\sigma_i^1(0,1) - \sigma_i^1(1,0))$ means that, starting with any positive term, we can match it with a negative term to cancel that term, which is paired with a postive term, having a matching negative term, etc., until the initial positive term is cancelled. This sub-chain represents a collection of paths which concatenate to a loop, so $z =$ (this loop) + (the remaining terms) . Induction implies that z can be written as a sum of (sums of paths forming loops), which is (as above) homologous to the sum of loops. Choosing paths from the start of these loops to our chosen basepoint (which is the only place where we use path connectedness, we can concatenate the based loops $\overline{\gamma} * \sigma * \gamma$ to a single based loop η, which under H is sent to a chain homologous to z. So $H[\eta] = [z]$.

Since $H_1(X)$ is abelian (and $\pi_1(X)$ need not be), the kernel of H contains the commutator subgroup $[\pi_1(X), \pi_1(X)]$. We now show that, if X is path connected, H induces an isomorphism $H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)]$. To show this, it remains to show that $\ker(H) \subseteq [\pi_1(X), \pi_1(X)]$. Or put differently, the ineduced map from $\pi_1(X)_{ab} = \pi_1(X)/[\pi_1(X), \pi_1(X)]$ (i.e., $\pi_1(X)$, written using additive notation) to $H_1(X)$ is injective. So suppose $[\gamma] \in \pi_1(X)$ and, thought of as a singular 1-simplex, $\gamma = \partial w$ for some 2-simplex $w = \sum a_i \sigma_i^2$. As before, we may assume that all $a_i = 1$, by reversing orientation and writing multiples as sums. By adding "tails" from each image of a vertex of each σ_i^2 to our chosen basepoint x_0, we may assume that the image of every face of Δ^2, under the σ_i , is a loop at x_0 (by essentially replacing each σ_i with a τ_i which first collapses little triangle at each vertex to arcs, maps the resulting central triangle via σ_i, and the arcs via the paths).

Once we have made this slight alteration, the equation $\gamma = \partial w = \displaystyle\sum_{i=1}^{n} \sum_{j=0}^{2} \partial_j \sigma_i = 0$ makes sense (and is true) in both ($C_1(X)$ hence $Z_1(X)$ hence) $H_1(X)$ and $\pi_1(X)_{ab}$, the first essentially by definition and the second because all of the $\partial_j \sigma_i$ are loops at x_0 and, in $\pi_1(X)$, $(\partial_0 \sigma_i)\overline{\partial_1 \sigma_i}(\partial_2 \sigma_i)$ is null-homotopic, so is trivial in $\pi_1(X)$. Written additively, this means that in $\pi_1(X)_{ab}$, $\partial_0 \sigma_i - \partial_1 \sigma_i + \partial_2 \sigma_i = 0$. So $\gamma = 0$ in $\pi_1(X)_{ab}$, as desired.

The Hurewicz map $H : \pi_1(X) \to H_1(X)$ induces, when X is path-connected, an isomorphism from $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ to $H_1(X)$. This result can be used in two ways; knowing a (presentation for) $\pi_1(X)$ allows us to compute $H_1(X)$, by writing the relators additively, giving $H_1(X)$ as the free abelian group on the generators, modulo the kernel of the "presentation matrix" given by the resulting linear equations. Conversely, knowing $H_1(X)$ provides information about $\pi_1(X)$. For example, a calculation on the way to invariance of domain implied that for every knot K in S^3 (i.e., the image of an embedding $h : S^1 \hookrightarrow S^3$), $H_1(S^3 \setminus K) \cong \mathbb{Z}$. This implies that the abelianization of $G_K = \pi_1(S^3 \setminus K)$ (i.e., the largest abelian quotient of G_K) is $\mathbb{Z}$. But this in turn implies that for every integer $n \geq 2$, there is a <u>unique</u> surjective homomorphism $G_K \to \mathbb{Z}_n$, since such a homomorphism must factor through the abelianization, and there is exactly one surjective homomorphism $\mathbb{Z} \to \mathbb{Z}_n$! Consequently, there is a unique (normal) subroup (the kernel of this homomorphism) $K_n \subseteq G_K$ with quotient $\mathbb{Z}_n$. Using the Galois correspondence, there is a (unique) covering space X_n of $X = S^3 \setminus K$ corresponding to K_n, called the n-fold cyclic covering of K . This space is determined by K and n, and so its homology groups are determined by the same data. And even though homology cannot distinguish between two knot complements, K, K', it might be the case that homology <u>can</u> distinguish between their cyclic coverings. Consequently, if $H_1(X_n) \ncong H_1(X'_n)$, then K and K' have non-homeomorphic complement, and so represent "different" embeddings, hence different knots. In practice, one can compute presentations for $\pi_1(X_n)$ (in several different ways), and so one can compute $H_1(X_n)$, providing an effective way to use homology to distinguish knots! This approach was ultimately formalized (by Alexander) into a polynomial invariant of knots, known as the Alexander polynomial.

Computing the homology of the cyclic coverings can be done in several ways. The Reidemeister-Schreier method will allow one to compute a presentation for the kernel of a homomorphism $\varphi : G \to H$, given a presentation of G and a *transversal* of the map, which is a representative of each coset of G modulo the kernel. Abelianizing this will give homology computation. Another approach uses *Seifert surfaces*, orientable surfaces with $\partial\Sigma = K$, to cut $S^3 \setminus K$ open along. Writing $S^3 \setminus K = (S^3 \setminus N(\Sigma)) \cup N(\Sigma)$ allows us to use Mayer-Vietoris to compute homology. But the cyclic covering spaces can be built by "unwinding" this view of $S^3 \setminus K$; instead of gluing the two ends of $N(K)$ to the same $S^3 \setminus N(\Sigma)$, we can take n copies of $S^3 \setminus N(\Sigma)$ and glue them together in a circle. Mayer-Vietoris again tells us how to compute the homology of the resulting space. Details may be found on the accompanying pages taken from Rolfsen's "Knots and Links".