Math 872 Problem Set 8

Starred (*) problems are due Thursday, April 16.

(*) 53. [Hatcher, p.132, #12] Show that for chain maps f, g between chain complexes $\mathcal{A} = \{A_n\}, \mathcal{B} = \{B_n\}$, the relation "$f$ and g are chain homotopic" is an equivalence relation.

54. [Hatcher, p.133, # 27(a)] Let $f : (X, A) \to (Y, B)$ be a map of pairs such that both $f : X \to Y$ and $f : A \to B$ are homotopy equivalences. Show that the induced map $f_* : H_n(X, A) \to H_n(Y, B)$ is an isomorphism for all n.

55. [Hatcher, p.132, #15] Show that if $A \subseteq X$, then the inclusion map $i : A \to X$ induces an isomorphism on homology groups $\Leftrightarrow H_n(X, A) = 0$ for all $n \geq 0$.

(*) 56. Show that if a short exact sequence $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *splits*, that is, there is a map $\gamma : B \to A$ with $\gamma \circ \alpha = I$, then the map $\varphi : B \to A \oplus C$ given by $b \mapsto (\gamma(b), \beta(b))$, is an isomorphism.

[This is part of the Splitting Lemma, proved in Hatcher, p.147. Splitting is equivalent to the existence of $\delta : C \to B$ satisfying $\beta \circ \delta = I$, but this is irrelevant to the question above.]

(*) 57. Show that if $A \subseteq X$ and $r : X \to A$ is a retraction, then for every n,

$H_n(X) \cong H_n(A) \oplus H_n(X, A)$.

[Hint: show that the (piece of) the long exact homology sequence

$H_n(A) \to H_n(X) \to H_n(X, A)$ is "really"

$0 \to H_n(A) \to H_n(X) \to H_n(X, A) \to 0$, and splits.]

58. [Hatcher, p.156, # 12] $S^1 \times S^1/[S^1 \times \{*\} \cup \{*\} \times S^1$ is homeomorphic to S^2. Show that the quotient map $q : S^1 \times S^1 \to S^2$ induces an isomorphism on H_2, so q is not null-homotopic. Show, conversely (using covering spaces) that any map $p : S^2 \to S^1 \times S^1$ is null-hmotopic.

59. Find examples of spaces and subspaces $A_0 \subseteq X_0$ and $A_1 \subseteq X_1$ so that $H_*(X_0) \cong H_*(X_1)$ and $H_*(A_0) \cong H_*(A_1)$, but $H_*(X_0, A_0) \ncong H_*(X_1, A_1)$. (If you want to make it more challenging, find examples with all of the spaces path-connected? Note that Problem #54 gives a hint on how not to solve this problem...)