Math 872 NOT Problem Set 10

There are no starred (*) problems; no problems are due.

73. Find the (reduced singular) homology groups of the Möbius band $M = [0,1]\times[0,1]/\{(0,t)\sim(1,1-t)\ :\ t\in[0,1]\}$, and compute the inclusion-induced maps $i_* : \widetilde{H}_k(\partial M)\to\widetilde{H}_k(M)$ from the bundary circle of M into M.

74. Use the computations from problem # 73 and the Mayer-Vietoris sequence to compute the (reduced) homology groups of the projective plane $\mathbb{R}P^2 = S^2/(x\sim -x)$, which can be viewed as the union of a Möbius band M and a 2-disk D^2 with their boundary circles glued together.

75. Use the computations from problem # 73 and the Mayer-Vietoris sequence to compute the (reduced) homology groups of the Klein bottle K^2, which can be viewed as two Möbius bands with their boundary circles glued together $K^2 = M_1 \cup_{\partial M_1=\partial M_2} M_2$.

76. [Hatcher, p.156, # 12] $S^1\times S^1/[S^1\times\{*\}\cup\{*\}\times S^1$ is homeomorphic to S^2. Show that the quotient map $q : S^1\times S^1\to S^2$ induces an isomorphism on H_2, so q is not null-homotopic. Show, conversely (using covering spaces) that any map $p : S^2\to S^1\times S^1$ <u>is</u> null-hmotopic.

    [The LEHS of the pair $(S^1\times S^1, S^1\times\{*\}\cup\{*\}\times S^1)$ will help with the first part.]

77. Show that if $A\subseteq X$ and the identity map $I : X\to X$ is homotopic to a map $f : X\to X$ with $f(X)\subseteq A$, then for every n, $H_n(A)\cong H_n(X)\oplus H_{n+1}(X,A)$.

    (So A has more "holes" than X does...) [Hint: f can be written as $\iota\circ g$ where g is f with its codomain restricted to A, and $\iota : A\to X$ is inclusion. The hypothesis tells us something about $(\iota\circ g)_*$...]

78. Find an example of a pair of graphs Γ_1, Γ_2, so that one is a finite sheeted covering of the other $p : \Gamma_1\to\Gamma_2$, but the induced map on homology $p_* : H_1(\Gamma_1)\to H_1(\Gamma_2)$ is <u>not</u> injective. [Hint: it actually almost never <u>is</u> injective...]

79. Show that if $\mathcal{C} = \{C_n, \partial_n\}$, $\mathcal{C}'$ and $\mathcal{C}''$ are chain complexes, and $f, g : \mathcal{C}\to\mathcal{C}'$ and $h, j : \mathcal{C}'\to\mathcal{C}''$ are chain maps, <u>and</u> f and g are chain homotopic and h and j are chain homotopic, <u>then</u> $h\circ f$ and $j\circ g$ are chain homotopic.

80. Show that if X is a space with $H_k(X;\mathbb{Z})\cong\mathbb{Z}_n$, then $H_{k+1}(X;\mathbb{Z}_n)\neq 0$.

    (Hint: look at the LEHS induced by the SES of coefficient groups

    $$0\to\mathbb{Z}\overset{\times n}{\to}\mathbb{Z}\to\mathbb{Z}_n\to 0\ .)$$