Math 872 Problem Set 1

Starred (*) problems are due Thursday, February 11.

(*) 1. If $x_0 \in A \subseteq X$ and $\pi_1(X, x_0) = \{1\}$, show that, for any space Y, if a cts map $f : A \to Y$ extends to a map $g : X \to Y$, then $f_* : \pi_1(A, x_0) \to \pi_1(Y, f(x_0))$ is the trivial map.

2. [Hatcher, p.38, #5] Show that $\pi_1(X, x_0) = \{1\}$ for every $x_0 \in X$ if and only if every cts map $f : S^1 \to X$ extends to a map $F : D^2 \to X$ (i.e., for some F, we have $F \circ \iota = f$).

(*) 3. [Hatcher, p.38, #2] Show that, in general, if $\alpha, \beta : I \to X$ are paths which are *homotopic rel endpoints*, $\alpha(0) = \beta(0) = x_0$, $\alpha(1) = \beta(1) = x_1$, then their associated change of basepoint maps are equal: $\Phi_\alpha = \Phi_\beta : \pi_1(X, x_0) \to \pi_1(X, x_1)$.

[N.B.: Hatcher calls these maps β_α and (sorry...) β_β, for reasons I don't understand. Maybe β = 'basepoint'?]

4. [Hatcher, p.38, #3] Show that for a path-connected space X, $\pi_1(X)$ is abelian $\Leftrightarrow$ the change of basepoint maps are all independent of path, i.e.,

for $\alpha, \beta : I \to X$ with $\alpha(0) = \beta(0) = x_0$ and $\alpha(1) = \beta(1) = x_1$, we always have $\Phi_\alpha = \Phi_\beta : \pi_1(X, x_0) \to \pi_1(X, x_1)$.

[Useful notation: our hypothesis can be expressed symbolically by saying that α, β are maps of triples; $\alpha, \beta : (I, 0, 1) \to (X, x_0, x_1)$.]

5. [Hatcher, p.38, #1] Show that homotopy of paths, rel endpoints, satisfies a 'cancellation law': if $\gamma_0, \gamma_1 : (I, 0, 1) \to (X, x_0, x_1)$ and $\eta_0, \eta_1 : (I, 0, 1) \to (X, x_1, x_2)$ satisfy $\gamma_0 * \eta_0 \simeq \gamma_1 * \eta_1$ rel ∂I <u>and</u> $\eta_0 \simeq \eta_1$ rel ∂I, then $\gamma_0 \simeq \gamma_1$ rel ∂I.

6. [Hatcher, p.39, # 13] If $x_0 \in A \subseteq X$ and A is path-connected, show that the inclusion-induced map $\iota_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective $\Leftrightarrow$ for every path $\gamma : I \to X$ with endpoints in A, $\gamma(0), \gamma(1) \in A$, γ is homotopic rel endpoints to a path in A, that is, $\gamma \simeq \alpha$ rel ∂I with $\alpha : I \to A \subseteq X$.

(*) 7. Our proof that homotopy equivalences induce isomorphisms on fundamental groups was <u>still</u> a little careless with basepoints. Show that if $f : X \to Y$ and $g : Y \to X$ are homotopy inverses, then with $x_0 \in X$, $y_0 = f(x_0)$, $x_1 = g(y_0)$, and $y_1 = f(x_1)$, $f_* : \pi_1(X, x_1) \to \pi_1(Y, y_1)$ surjective (from $f \circ g \simeq Id_Y$) implies that $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is surjective. [Hint: we know that there is a path from x_0 to x_1; the homotopy $g \circ f \simeq Id_X$ gives us one.]