Math 872 Problem Set 2

Starred (*) problems were due Thursday, February 18.

8. Show that every homomorphism $\varphi : \mathbb{Z} \to \pi_1(X, x_0)$ can be realized as the induced homomorphism $\varphi = f_*$ of a continuous map $f : (S^1, (1,0)) \to (X, x_0)$. [Hint: Look at $\varphi(1) \in \pi_1(X, x_0)$.]

9. A *topological group* is a space G with continuous maps $G \times G \to G$ and $G \to G$, denoted $(g, h) \mapsto g \cdot h$ and $g \mapsto g^{-1}$, which (together with an $e \in G$) make G a group. Show that for loops $\alpha, \beta : (I, \partial I) \to (G, e)$, the loop $\gamma(t) = \alpha(t) \cdot \beta(t)$ is homotopic, rel endpoints, to both $\alpha * \beta$ and $\beta * \alpha$. Conclude that for any topological group G, $\pi_1(G, e)$ is abelian.

   [Hint: $\alpha * \beta(t) = (\alpha * c_e)(t) \cdot (c_e * \beta)(t)$.]

(*) 10. [Hatcher p.40, #20] If $H : X \times I \to X$ is a homotopy with $H_0 = H_1 =$ the identity map, show that $\gamma(t) = H(x_0, t)$ is a loop representing an element $g = [\gamma] \in \pi_1(X, x_0)$ lying in the center of $\pi_1(X, x_0)$., i.e., $gh = hg$ for all $h \in \pi_1(X, x_0)$

11. [Hatcher, p.39, #10] The isomorphism $(\mathrm{pr}_X)_* \times (\mathrm{pr}_Y)_* : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ implies that for loops γ mapping into $X \times \{y_0\}$ and η mapping into $\{x_0\} \times Y$, we ust have $\gamma * \eta \simeq \eta * \gamma$ rel endponts (i.e., the two elements of $\pi_1(X \times Y, (x_0, y_0))$ commute). Build an explicit homotopy between the two loops.

(*) 12. Show that $\langle a, b \mid a^5 = b^4, aba = bab \rangle$ is a(nother) presentation for the trivial group (that is, show that the relations imply that $a = 1$ and $b = 1$).

13. (a) Show that the group $\langle a \mid a^n \rangle$ is isomorphic to the cyclic group $\mathbb{Z}_n$ on n elements.

   (b) Show that the group $G = \langle a, b, c \mid a^2, b^2, c^2, (ab)^3, (ac)^2, (bc)^3 \rangle$ is isomorphic to the symmetric group S_4 on four letters. [You can/should model this on our demonstration for S_3; find a surjective homomorphism to S_4, and show that G has at most 24 elements!]

(*) 14. Show that if $G = \langle A \mid R \rangle$, $x \notin A$ is a 'new' letter, and w is a word in the alphabet A, then $H = \langle A \cup \{x\} \mid R \cup \{xw\} \rangle$ is isomorphic to G. That is, build homomorphisms $G \to H$ and $H \to G$ that are inverses of one another.