Math 872 Problem Set 3

Starred (*) problems are due Thursday, February 25.

(*) 15. [Hatcher, p.52, #1] Show that the free product $G * H$ of non-trivial groups has trivial center; if $x \in G * H$ and $xy = yx$ for every $y \in G * H$, then $x = 1$. [Hint: show that if $x \neq 1$ then there is some y with $xy \neq yx$!]

16. What familiar group G is isomorphic to the group with presentation $\langle a, b, c, d \mid a^2, b^2, c^2, abc, bcd \rangle$? Find an explicit isomorphism. [N.B. Problem #14 can help you 'simplify' the presentation before deciding what G should be.]

17. Show that the groups $G = \langle a, b \mid aba = bab \rangle$ and $H = \langle x, y \mid x^2 = y^3 \rangle$ are isomorphic. That is, find maps $F(a, b) \to H$ and $F(x, y) \to G$ that induce homomophisms $G \to H$ and $H \to G$ that are inverses of one another! [These are two 'standard' presentations for the 3-strand braid group.]

(*) 18. Find a presentation for the fundamental group of the space X obtained by gluing two copies of $S^1 \times S^1$ together, where the circle $S^1 \times \{*\}$ of the first is glued to $\{*\} \times S^1$ of the second, via the map $(x, *) \mapsto (*, x)$. [It would probably help to reconstruct X as a bouquet of circles, with 2-disks attached.]

19. Show that if $x_0 \in X$ and $A \subseteq X$ is the *path component* of X containing x_0 - that is, A is the union of all path-connected subsets of X containing x_0 (note that since their common intersection is non-empty, this union is path-connected) - then the inclusion $\iota : A \to X$ induces an isomorphism $\iota_* : \pi_1(A, x_0) \to \pi_1(X, x_0)$. [Moral: π_1 can only "see" the parts of X that can be reached from x_0 by paths...]

20. For $n, m \in \mathbb{N}$, let $X_{m,n}$ be the space obtained by gluing two 2-disks to a circle, on using the map $\partial D^2 \to S^1$ given by $z \mapsto z^n$ (stealing from complex variables) and the other by the map $z \mapsto z^m$. Show that $\pi_1(X_{m,n})$ is cyclic, of order $\gcd(m, n)$.

(*) 21. [Hatcher, p.53, #6] Show that if $f : \partial D^n \to X$ is the attaching map of an n-cell D^n, with $n \geq 3$, then the inclusion $X \hookrightarrow X \cup_f D^n$ induces an isomorphism on π_1. Show that the same is true if we attach any (finite or infinite) collection of $n \geq 3$ cells.

22. Compute (i.e., find a presentation for) the fundamental group of the complement X of the coordinate axes in $\mathbb{R}^3$, i.e., $X = \{(x, y, z) \in \mathbb{R}^3 \ : \ |xy| + |xz| + |yz| \neq 0\}$.

    [Hint: Find a 'simpler' space, homotopy equivalent to X, to compute the fundamental group of.]