Math 872 Problem Set 3

Starred (*) problems are due Thursday, March 4.

23. The notion of a "universal mapping property" usually implies that there is only one object, up to isomorphism, which satisfies the property. Show that there is, for groups G and H, only one group "$G*H$" (and homomorphisms $\iota_G : G \to G*H$ and $\iota_H : H \to G*H$) satisfying the UMP: if $\varphi : G \to K$ and $\psi : H \to K$ are homomorphisms then there is a unique homomorphism $\omega : G*H \to K$ so that $\omega \circ \iota_G = \varphi$ and $\omega \circ \iota_H = \psi$. [Hint: given two such groups $(G*H)_1$ and $(G*H)_2$, use each one in turn as K ! And then compose your maps...]

(*) 24. Find a cell structure for, and compute a presentation for the fundamental group of, the space X obtained by gluing two Möbius bands $I \times I/\{(t,0) \sim (1-t,1) \ : \ t \in I\}$ along their boundary circles.

25. [Hatcher, p.53, #7] Let X be the quotient space formed from the 2-sphere S^2 by identifying the north and south poles. Put a cell structure on X and use this to compute $\pi_1(X)$.

(*) 26. [Hatcher, p.54, # 14 (sort of)] Let $X =$ the space obtained from a cube $J^3 = J \times J \times J$, $J = [-1,1]$, by gluing opposite square faces to one another with a 90-degree righthand twist (e.g., glue $J \times J \times \{0\}$ to $J \times J \times \{1\}$ by the map $(x,y,0) \mapsto (y,-x,1)$). Describe a CW structure for X and compute a presentation for $\pi_1(X)$. [The fun part: how many <u>different</u> 0-cells, 1-cells, etc. are there?]

27. [Hatcher, p.53, # 11 (sort of)] For a map $f : X \to X$, the *mapping torus* T_f of f is the space $X \times I/\{(x,0) \sim (f(x),1) \ : \ x \in X\}$ obtained by gluing the ends of the 'cylinder' $X \times I$ together using f. Find a presentation of $\pi_1(T_f)$ in terms of $f_* : \pi_1(X,x_0) \to \pi_1(X,x_0)$ in the case when $X = S^1 \vee S^1$ (joined along the basepoint x_0), and $f(x_0) = x_0$. [Hint: treating T_f as $X \vee S^1$ with cells attached can streamline this.]

28. For a CW complex X, show that if the 1-skeleton $X^{(1)}$ of X is path-connected, then X is path connected.

29. [Hatcher, p.19, # 14] Given $v, e, f > 0$ with $v - e + f = 2$, build a cell structure on S^2 with v 0-cells, e 1-cells, and f 2-cells.

(*) 30. [Hatcher, p.20, #28] Show that if the pair (X,A) satisfies the homotopy extension property, then for any map $f : A \to Y$ the pair (Z,Y), where $Z = X \coprod Y/\{a \sim f(a) \ : \ a \in A\}$ is the space built by gluing X to Y along A using f, also satisfies the homotopy extension property.

31. [Hatcher, p.20 #22] Show that if X is a CW complex, $A, B \subseteq X$ are subcomplexes, $A \cup B = X$, and A, B and $A \cap B$ are contractible, then X is contractible.