Math 872 Problem Set 6

Starred (*) problems are due Thursday, March 18.

40. Let $p : \widetilde{X} \to X$ be a finite-sheeted covering map. Show that $\widetilde{X}$ is compact and Hausdorff $\Leftrightarrow$ X is compact and Hausdorff.

41. Let $Y \subseteq \mathbb{R}^2$ denote the *quasi-circle*, given by $Y =$

$$\{(x, \sin(1/x)) : x \in (0, \tfrac{1}{\pi}]\} \cup \{0\} \times [-1, 1] \cup [-\tfrac{1}{\pi}, 0] \times \{0\} \cup \{(\tfrac{1}{\pi}\cos(t), -\tfrac{1}{\pi}\sin(t)) 0 \leq t \leq \pi\}$$

(See Hatcher, p.79, problem # 7 for an (approximate) picture.) The quotient map

$q : Y \to Y/\sim$ that collapses the vertical subinterval $\{0\} \times [-1, 1]$ to a point gives a space homeomorphic to S^1. Show that $\pi_1(Y) = \{1\}$ (hint: show that any path in Y is disjoint from $\{(x, y) \in Y \ : \ 0 < x < \epsilon\}$ for some $\epsilon > 0$), but the map q does not lift to the universal covering $p : \mathbb{R} \to S^1$. [This demonstrates that we cannot in general eliminate the hypothesis that Y be locally path-connected, in the lifting criterion.]

(*) 42. [Hatcher, p.79, # 8] Let $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be covering spaces of path-connected, locally path connected spaces X and Y with $\widetilde{X}$ and $\widetilde{Y}$ simply-connected. Show that if X and Y are homotopy equivalent, then $\widetilde{X}$ and $\widetilde{Y}$ are homotopy equivalent.

[Note: the problem statement in Hatcher points to an earlier problem that might help...]

43. [Hatcher, p.80, problem # 15] Suppose $p : \widetilde{X} \to X$ is a simply connected covering space of X, and $A \subseteq X$ is a path-connected, locally path-connected subspace of X, with $\widetilde{A} \subseteq \widetilde{X}$ a path component of $p^{-1}(A)$. Show that $p_*(\pi_1(\widetilde{A})) \subseteq \pi_1(A)$ is the kernel of the inclusion-induced homomorphism $\pi_1(A) \to \pi_1(X)$. [Hint: which loops lift to loops?]

(*) 44. Construct a finite-sheeted covering space of the wedge of two circles $S^1 \vee S^1$ (with fundamental group $F(a, b)$ in the 'standard way') so that a loop representing $abab^{-1}$ lifts to a loop at some basepoint but a loop representing $baab$ does <u>not</u>. [N.B. There are <u>tons</u> of distinct answers!]

45. Show that the fundamental group of the closed orientable surface Σ_2 of genus 2 is not abelian.

[Hint: to show that for some pair of loops γ, η we have that $\gamma * \eta * \overline{\gamma} * \overline{\eta}$ isn't homotopically trivial, you need to show that it (at least once!) does not lift to a loop in <u>some</u> covering space of Σ_2. You can choose any basepoint you want; they give isomorphic groups.]

46 [Hatcher, p.79, problem # 9] Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $f : X \to S^1$ is homotopic to a constant map.

(*) 47. If $p : \widetilde{X} \to X$ is a covering map, $x_0 \in X$, and $\widetilde{x}_0, \widetilde{x}_1 \in p^{-1}(\{x_0\})$, <u>and</u> η is a path in $\widetilde{X}$ from $\widetilde{x}_0$ to $\widetilde{x}_1$ with $[p \circ \eta] = g \in \pi_1(X, x_0)$, show that the subgroups $p_*(\pi(\widetilde{X}, \widetilde{x}_0)) = H$ and $p_*(\pi(\widetilde{X}, \widetilde{x}_1)) = K$ are <u>conjugate</u>; $H = gKg^{-1}$.