# Math 872 Problem Set 7

Starred (*) problems are due Thursday, March 25.

(**) 48. A group $G$ is called *residually finite* if for every $g \in G$ with $g \neq 1$, there is a finite group $H$ and a homomorphism $\varphi : G \to H$ with $\varphi(g) \neq 1$. Show that $G$ is residually finite if and only if for some (equivalently, any!) CW-complex $X$ with $\pi_1(X, x_0) \cong G$ and any loop $\gamma : I \to X$ at $x_0$ with $1 \neq [\gamma] \in \pi_1(X)$, there is a finite-sheeting covering space $p : \widetilde{X} \to X$ and basepoint $\widetilde{x_0}$ over $x_0$ so that $\gamma$ does not lift to a loop at $\widetilde{x_0}$.

49. [Hatcher, p.79, problem # 4, sort of] Build a connected, simply connected covering space $p : \widetilde{X} \to X$ of the wedge of a circle and a 2-sphere $X = S^1 \vee S^2$ .

[Note: problem # 43 might give you some guidance on what parts of it should look like...] [Or: what is the fundamental group of $X$? Your covering should have that for its deck transformation group.]

50. The 'pseudo-projective plane(s)' $P_n$ are the spaces built by gluing a 2-disk onto $S^1$ using the attaching map $t \mapsto e^{2\pi int}$ (i.e., wrapping around $S^1$ $n$ times). Describe (a CW structure on) the universal covering $\widetilde{P}_n$ of $P_n$, and what the deck transformations of the universal covering look like. [Hint: we know $\pi_1(P_n)$, so we know how many sheets the covering should have.]

51. Is the subgroup $H$ of $F(a, b)$ generated by the elements $\{aba, bba, bab, abb\} \subseteq F(a, b)$ isomorphic to a free group on 4 letters? That is, if you build a covering space $p : \widetilde{X} \to S^1 \vee S^1$ with $p_*(\pi_1(\widetilde{X})) = H$ (and you should!), is $\pi_1(\widetilde{X})$ free on four generators?

(**) 52. Show that for $G = \langle a, b \mid abab^{-1}a, abbab\rangle$ that the subgroup $H = \langle a\rangle$ has finite index in $G$, by (using the presentation complex $X_G$ for G) building the covering projection $\widetilde{X}^{(1)} \to X_G^{(1)}$ for the covering space correseponding to $H$, via folding. Is $H$ a normal subgroup of $G$?

53. Repeat the analysis for Problem #52, except for the subgroup $H = \langle b\rangle$.

54. Show that if a group $G$ acts freely ($x = gx \Rightarrow g = 1$) and properly discontinuously (for all $x \in X$ there is a nbhd $\mathcal{U}$ of $x$ such that $\{g : g(\mathcal{U}) \cap \mathcal{U} \neq \emptyset\}$ is finite) on a space $X$, then the quotient map $p : X \to X/G = X/\{x \sim gx \text{ for all } g \in G\}$ given by $p(x) = [x]$ is a covering map. In particular if $X$ is Hausdorff and $G$ is a finite group acting freely on $X$, then $p : X \to X/G$ is a covering map.

[Pointless remark: some people would write our quotient space as $G\backslash X$, since $G$ is acting on the left, and so is being quotiented out from the left, although the Wikipedia entry on the matter, *http://en.wikipedia.org/wiki/Group_action*, agrees with us in this. Besides, as one usually learns when TeXing things, TeX doesn't like \ as a symbol, it asks what the macro "\[next symbol]" is supposed to mean ...]

(**) 55. [Hatcher, p.82, # 32] ("Everything is really captured by the 1-skeleton") If $X$ is a CW complex and $p : \widetilde{X} \to X$ is a covering map, we can give $\widetilde{X}$ the 'induced' CW structure (lift characteristic maps). This gives (by restriction) covering maps $p : \widetilde{X}^{(k)} \to X^{(k)}$ . Show that two covering maps $p : \widetilde{X}_1 \to X$ and $p : \widetilde{X}_2 \to X$ are isomorphic if and only if the induced maps $p : \widetilde{X}_1^{(1)} \to X^{(1)}$ and $p : \widetilde{X}_2^{(1)} \to X^{(1)}$ are isomorphic. [Note: for ($\Leftarrow$) you already have the maps $p$ and $q$; you can build the homeomorphism $\widetilde{X}_1 \to \widetilde{X}_2$ inductively, on $k$-skeletons (by'lifting' the attaching maps for $\widetilde{X}_1^{(k)}$).]