Math 872 Problem Set 9

Starred (*) problems are due Thursday, April 22.

65. Regarding the n-simplex $X = \Delta^n$ as a Δ-complex in the natural way, show that if $A \subseteq X$ is a subcomplex with $H^{\Delta}_{n-1}(A) \neq 0$, then $A = \partial\Delta^n$.

    [Hint: show that an $(n-1)$-cycle for A (and hence for X) must either be 0 or have non-zero coefficient for <u>every</u> $(n-1)$-dimensional face of X.]

(**) 66. [Hatcher, p.132, #11] Show that if $A \subseteq X$ is a retract of X (recall this means there is a cts map $r : X \to A$ with $r(a) = a$ for all $a \in A$), then the inclusion map $\iota : A \to X$ induces an injection on all singular homology groups.

67. [Hatcher, p.132, #12] Show that for chain maps f, g between chain complexes $\mathcal{A} = \{A_n\}, \mathcal{B} = \{B_n\}$, the relation "$f$ and g are chain homotopic" is an equivalence relation.

(**) 68. Show that if a short exact sequence $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$ *splits*, that is, there is a map $\gamma : B \to A$ with $\gamma \circ \alpha = I$, then the map $\varphi : B \to A \oplus C$ given by $b \mapsto (\gamma(b), \beta(b))$, is an isomorphism.

    [This is part of the Splitting Lemma, proved in Hatcher, p.147. Splitting is equivalent to the existence of $\delta : C \to B$ satisfying $\beta \circ \delta = I$, but this is irrelevant to the question above.]

69. [Hatcher, p.132, # 14] Is there a short exact sequence $0 \to \mathbb{Z}_4 \to \mathbb{Z}_8 \oplus \mathbb{Z}_2 \to \mathbb{Z}_4 \to 0$? (That is, do homomorphisms exist that would make this sequence of groups exact?)

(**) 70. [Hatcher, p.132, #15] Show that if $A \subseteq X$, then the inclusion map $i : A \to X$ induces an isomorphism on all homology groups $\Leftrightarrow H_n(X, A) = 0$ for all $n \geq 0$. (See the problem statement in Hatcher for some guidance on proving this...)

71. Show that if $A \subseteq X$ and $r : X \to A$ is a retraction, then for every n,

    $H_n(X) \cong H_n(A) \oplus H_n(X, A)$.

    [Hint: show that the (piece of) the long exact homology sequence

    $H_n(A) \to H_n(X) \to H_n(X, A)$ is "really"

    $0 \to H_n(A) \to H_n(X) \to H_n(X, A) \to 0$, and splits.]

72. [Hatcher, p.132, #16] (a) Show that $H_0(X, A) = 0 \Leftrightarrow A$ meets every path component of X.

    (b) Show that $H_1(X, A) = 0 \Leftrightarrow \iota_* : H_1(A) \to H_1(X)$ is surjective and each path-component of X contains at most one path-component of A.