The zen of long exact sequences

Extracting information from a long exact sequence amounts to exploiting knowledge of the identities of some of the groups and/or some of the maps involved. The general setup we find coming from the conversion of short exact sequences of chain maps to long exact sequences in homology is that every third group comes from the same chain complex. So we generally expect things to come in threes; we know/control every third group (when we control one) and/or every third map (when we control one).

A typical situation is that we are building a new space/chain complex out of two others that we "understand", meaning that we know the groups for two of the three spaces and the maps running between them. In the long exact sequence

$$\longrightarrow A_n \xrightarrow{\alpha_n} B_n \xrightarrow{\beta_n} C_n \xrightarrow{\gamma_n} A_{n-1} \xrightarrow{\alpha_{n-1}} B_{n-1} \longrightarrow$$

if we know the groups A_k and B_k and the maps α_k between them, then we can 'harvest' the long exact sequence to give a collection of short exact sequences by noting that Ker $\beta_n =$ Im α_n (which we know) and Im $\gamma_n =$ Ker α_{n-1} (which we know). So γ_n maps onto Ker α_{n-1}, making

$$A_n \xrightarrow{\alpha_n} B_n \xrightarrow{\beta_n} C_n \xrightarrow{\gamma_n} \text{Ker } \alpha_{n-1} \longrightarrow 0$$

exact; and, on the other end, since by one of those Noether theorems, the induced map $\overline{\beta}_n : B_n/\text{Ker } \beta_n \to C_n$ is injective (with the same image as β_n), we have

$$0 \longrightarrow B_n/\text{Im } \alpha_n \xrightarrow{\overline{\beta}_n} C_n \xrightarrow{\gamma_n} \text{Ker } \alpha_{n-1} \longrightarrow 0$$

is (short) exact. The groups on the ends are ones that, in principle, we know how to compute from the data $A_n \xrightarrow{\alpha_n} B_n$. On the other hand, knowledge of the end groups is not enough to determine the middle group C_n; but if we know that the sequence 'splits', as described in Hatcher's book and our problem set(s) (which is always true, for example, if the rightmost group Ker α_{n-1} is free abelian (which, in turn, is true, for example, if A_{n-1} is free abelian)), then the middle group is the direct sum of the outer groups. Otherwise, we need to 'know' something about one of the maps.

A similar analysis applies when we 'know' $B_n \xrightarrow{\beta_n} C_n$, or we know $C_n \xrightarrow{\gamma_n} A_{n-1}$.

Another situation that we can often engineer is to know that every third map (the α_k, for example) is always injective, or always surjective, or always the 0 map. [We will encounter topological conditions which make this happen, e.g., for the long exact sequence of a pair, in the presence of a retraction $r : X \to A$ for $A \subseteq X$.] As it happens, each of these sort of imply the other, in fact: in the exact sequence

$$\longrightarrow B_{n+1} \xrightarrow{\beta_{n+1}} C_{n+1} \xrightarrow{\gamma_{n+1}} A_n \xrightarrow{\alpha_n} B_n \xrightarrow{\beta_n} C_n \xrightarrow{\gamma_n} A_{n-1} \longrightarrow$$

if α_n is injective, then $\{0\} =$ Ker $\alpha_n =$ Im γ_{n+1}, so γ_{n+1} is the 0 map, so Ker $\gamma_{n+1} = C_{n+1} =$ Im β_{n+1}, so β_{n+1} is surjective. Similarly, if α_n is surjective, then β_n is the 0 map, and so γ_n is injective, while if α_n is the 0 map, then β_n is injective and γ_{n+1} is surjective. So if all of the α_k have the same character, then every third map (somewhere) in the long exact sequence is the 0 map, and so the map that precedes it is surjective and the map that follows it is injective. This enables us, again, to harvest the long exact sequence to create a collection of short exact sequences (since a 0-map might as well map to the 0 group, while on the other end, an injective map might as well be preceded by the 0-map from the 0 group!), which will be one of

$$0 \longrightarrow A_n \xrightarrow{\alpha_n} B_n \xrightarrow{\beta_n} C_n \longrightarrow 0 \text{ , or}$$

$$0 \longrightarrow C_{n+1} \xrightarrow{\gamma_{n+1}} A_n \xrightarrow{\alpha_n} B_n \longrightarrow 0 \text{ , or}$$

$$0 \longrightarrow B_n \xrightarrow{\beta_n} C_n \xrightarrow{\gamma_n} A_{n-1} \longrightarrow 0$$

If we treat the A_n and B_n as 'known', and the C_n as unknown, then in the first case $C_n \cong B_n/\text{Im } \alpha_n$, while in the second case $C_{n+1} \cong \text{Im } \gamma_{n+1} \cong \text{Ker } \alpha_n$. The third case, again, poses an 'extension problem', and so without furher information about B_n, A_{n-1}, and the maps we cannot definitively determine C_n.

And recognizing a 0 map can happen many ways. One of the groups being 0 is the quickest, but, for example, the only map from a torsion group (all elements have finite order) to a torsion-free group (all non-zero elements have infinite order!) is the 0 map.