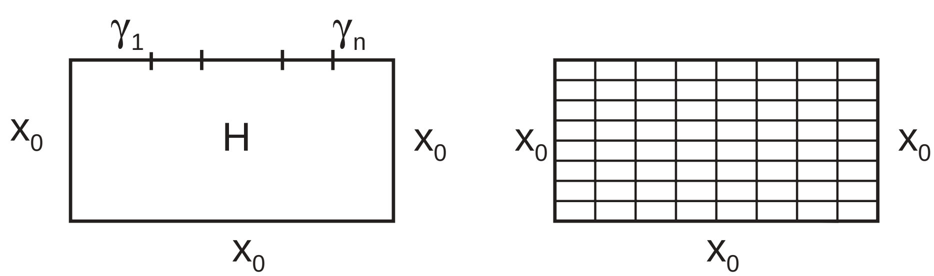
Math 971 Algebraic Topology

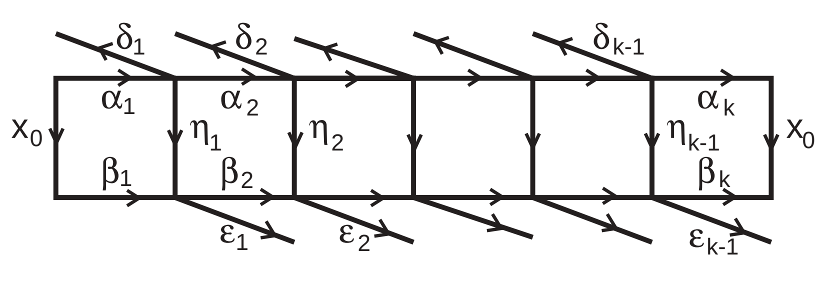
February 3, 2005

We now turn our attention to proving Seifert - van Kampen; understanding the kernel of the map $\phi : \pi_1(X_1) * \pi_1(X_2) \rightarrow \pi_1(X)$, under the hypotheses that X_1, X_2 are open, $A = X_1 \cap X_2$ is path-connected, and the basepoint $x_0 \in A$. So we start with a product $g = g_1 \cdots g_n$ of loops alternately in X_1 and X_2, which when thought of in X is null-homotopic. We wish to show that g can be expressed as a product of conjugates of elements of the form $i_{1*}(a)(i_{2*}(a))^{-1}$ (and their inverses). The basic idea is that a "big" homotopy can be viewed as a large number of "little" homotopies, which we essentially deal with one at a time, and we find out how little "little" is by using the same Lebesgue number agument that we used before.

Specifically, if H is the homotopy, rel basepoint, from $\gamma_1 * \cdots * \gamma_n$, where γ_i is a based loop representing g_i, and the constant loop, then, as before, $\{H^{-1}(X_1), H^{-1}(X_2)\}$ is an open cover of $I \times I$, and so has a Lebesgue number ϵ. If we cut $I \times I$ into subsquares, with length $1/N$ on a side, where $1/N < \epsilon$, then each subsquare maps into either X_1 or X_2. The idea is to think of this as a collection of horizontal strips, each cut into squares. Arguing by induction, starting from the bottom (where our conclusion will be obvious), we will argue that if the bottom of the strip can be expressed as an element of the group
$N =< i_{1*}(\gamma)(i_{2*}(\gamma))^{-1} : \gamma \in \pi_1(A) >^N \subseteq \pi_1(X_1) * \pi_1(X_2)$
(i.e., as a product of conjugates of such loops), then so can the top of the strip.



And to do this, we work as before. We have a strip of squares, each mapping into either X_1 or X_2. If adjacent squares map into the same subpace, amalgamate them into a single larger rectangle. Continuing in this way, we can break the strip into subrectangles which alternately map into X_1 or X_2. This means that the vertical arcs in between map into $X_1 \cap X_2 = A$, and represent paths η_i in A. Their endpoints also map into A, and so can be joined by paths (δ_i on the top, ϵ_i on the bottom) in A to the basepoint. The top of the strip is homotopic, rel basepoint, to
$(\alpha_1 * \delta_1) * (\overline{\delta_1} * \alpha_2 * \delta_2) * \cdots * (\overline{\delta_{k-1}} * \alpha_k)$
each grouping mapping into either X_1 or X_2. The rectangles demonstrate that each grouping is homotopic, rel basepoint, to the product of loops
$(\overline{\delta_i} * \eta_i * \epsilon_i) * (\overline{\epsilon_i} * \beta_i * \epsilon_{i+1}) * (\overline{\epsilon_{i+1}} * \overline{\eta_{i+1}} * \delta_{i+1}) = a_i b_i a_{i+1}^{-1}$
where this is thought of as a product in either $\pi_1(X_1)$ or $\pi_1(X_2)$. The point is that when strung together, this appears to give $(b_1 a_2^{-1})(a_2 b_2 a_3^{-1}) \cdots (a_k b_k)$, with lots of cancellation, but in reality, the terms $a_i^{-1} a_i$ represent elements of N, since the two "cancelling" factors are thought of as living in the different groups $\pi_1(X_1), \pi_1(X_2)$. The remaining terms, if we delete these "cancelling" pairs, is $b_1 \cdots b_k = \beta_1 * \epsilon_1 * \cdots * \overline{\epsilon_i} * \beta_i * \epsilon_{i+1} * \cdots * \overline{\epsilon_k} * \beta_k$, which is homotopic rel endpoints to $\beta_1 * \cdots * \beta_k$, which, by induction, can be represented as a product which lies in N.



So, we can obtain the element represented by the top of the strip by inserting elements of N into the bottom, which is a word having a representation as an element of N. The final problem to overcome is that the insertions represented by the vertical arcs might not be occuring where we want them to be! But this doesn't matter; inserting a word w in the middle of another uv (to get uwv) is the same as multiplying uv by a conjugate of w; $uwv = (uv)(v^{-1}wv)$, so since the bottom of the strip is in N, and we obtain the top of the strip by inserting elements of N into the bottom, the top is represented by a product of conjugates of elements of N, so (since N is normal) is in N. And a final final point; the subrectangles may not have cut the bottom of the strip up into the same pieces that the inductive hypothesis used to express the bottom as an element of N. It didn't even cut it into loops; we added paths at the break points to make that happen. The inductive hypothesis would have, in fact, added its own extra paths, at possibly different points! But if we add both sets of paths, and cut the loop up into even more pieces, then we end up with a loop, which we have expressed as a product in $\pi_1(X_1) * \pi_1(X_2)$ in two (possibly different) ways, since the two points of view will have interpreted pieces as living in different subspaces. But when this happens, it must be because the subloop really lives in $X_1 \cap X_2 = A$. Moving from one to the other amounts to repeatedly changing ownership between the two sets, which in $\pi_1(X_1) * \pi_1(X_2)$ means inserting an element of N into the product (that is literally what elements of N do). But as before, these insertions can be collected at one end as products of conjugates. So if one of the elements is in N, the other one is, too.

Which completes the proof!