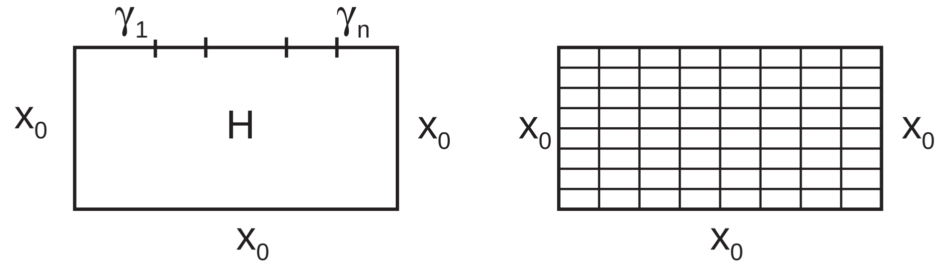
Math 971 Algebraic Topology

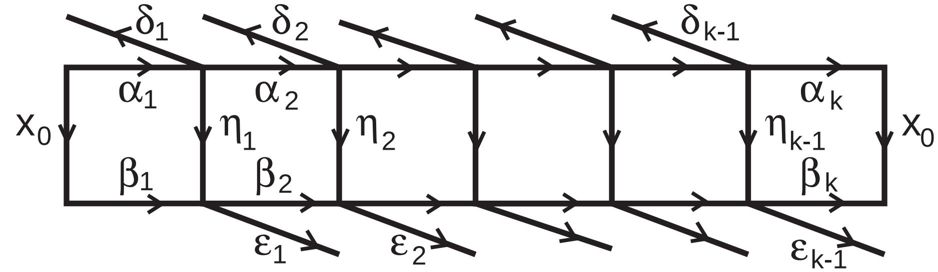
February 3, 2005

Understanding the kernel of the map $\phi : \pi_1(X_1) * \pi_1(X_2) \to \pi_1(X)$, where X_1, X_2 are open, $A = X_1 \cap X_2$ is path-connected, and the basepoint $x_0 \in A$. Start with $g = g_1 \cdots g_n$ a product of loops alternately in X_1 and X_2, which when thought of in X is null-homotopic. Want to show that g is a product of conjugates of elements of the form $i_{1*}(a)(i_{2*}(a))^{-1}$ (and their inverses). The basic idea: walk through the null-homotopy a little bit at a time!



If H is the homotopy, rel basepoint, from $\gamma_1 * \cdots * \gamma_n$, where γ_i is a based loop representing g_i, and the constant loop, then $\{H^{-1}(X_1), H^{-1}(X_2)\}$ is an open cover of $I \times I$, and so has a Lebesgue number ϵ. If we cut $I \times I$ into subsquares, with length $1/N$ on a side, where $1/N < \epsilon$, then each subsquare maps into either X_1 or X_2. We will think of this as a collection of horizontal strips, each cut into squares. Arguing by induction, starting from the bottom (where our conclusion will be obvious), we will argue that if the bottom of the strip can be expressed, in $\pi_1(X_1) * \pi_1(X_2)$, as an element of the group $N =< i_{1*}(\alpha)(i_{2*}(\alpha))^{-1} : \gamma \in \pi_1(A) >^N$ (i.e., as a product of conjugates of such loops), then so can the top of the strip.

So we have a strip of squares, each square mapping into either X_1 or X_2. Amalgamating squares if they map into the same subset, we can break the strip into subrectangles which alternately map into X_1 or X_2. This means that the vertical arcs in between map into $X_1 \cap X_2 = A$, and represent paths η_i in A. Their endpoints also map into A, and so can be joined by paths (δ_i on the top, ϵ_i on the bottom) in A to the basepoint. The top of the strip is homotopic, rel basepoint, to $(\alpha_1 * \delta_1) * (\overline{\delta_1} * \alpha_2 * \delta_2) * \cdots * (\overline{\delta_{k-1}} * \alpha_k)$ each grouping mapping into either X_1 or X_2. The subrectangles demonstrate that each grouping is homotopic, rel basepoint, to the product of loops $(\overline{\delta_i} * \eta_i * \epsilon_i) * (\overline{\epsilon_i} * \beta_i * \epsilon_{i+1}) * (\overline{\epsilon_{i+1}} * \overline{\eta_{i+1}} * \delta_{i+1}) = a_i b_i a_{i+1}^{-1}$ where this is thought of as a product in either $\pi_1(X_1)$ or $\pi_1(X_2)$. The point is that when strung together, this appears to give $(b_1 a_2^{-1})(a_2 b_2 a_3^{-1}) \cdots (a_k b_k)$, with lots of cancellation, but in reality,



the terms $a_i^{-1} a_i$ represent elements of N, since the two "cancelling" factors really living in the different groups $\pi_1(X_1), \pi_1(X_2)$. The remaining terms, after "cancelling" these pairs, is $b_1 \cdots b_k = \beta_1 * \epsilon_1 * \cdots * \overline{\epsilon_i} * \beta_i * \epsilon_{i+1} * \cdots * \overline{\epsilon_k} * \beta_k$, which is homotopic rel endpoints to $\beta_1 * \cdots * \beta_k$, which, by induction, can be represented as a product which lies in N.

So the top of the strip is obtained by inserting elements of N into the bottom, which is a word having a representation as an element of N. But inserting a word w in the middle of another uv (to get uwv) is the same as multiplying uv by a conjugate of w; $uwv = (uv)(v^{-1}wv)$, so since the bottom of the strip is in N, the top is represented by a product of an element of N with conjugates of elements of N, so (since N is normal) is in N. But! Our subrectangles may not have cut the bottom of the strip up into the same pieces that we had been told to use to express the bottom as an element of N. They didn't even cut it into loops; we added paths at the break points to make that happen. An inductive hypothesis would have, in fact, added its own extra paths, at possibly different points! But if we add both sets of paths, and cut the loop up into even more pieces, then we end up with a loop, which we have expressed as a product in $\pi_1(X_1) * \pi_1(X_2)$ in two ways, since the two points of view (the one we started with and the one we have just imposed working down from above) will have interpreted pieces as living in different subspaces. But when this happens, it must be because the subloop really lives in $X_1 \cap X_2 = A$. Moving from one point of view to the other amounts to repeatedly changing ownership between the two sets, which in $\pi_1(X_1) * \pi_1(X_2)$ means inserting an element of N into the product (that is literally what elements of N do). But as before, these insertions can be collected at one end as products of conjugates. So if one of the representations is in N, the other one is, too.