Math 971 Algebraic Topology

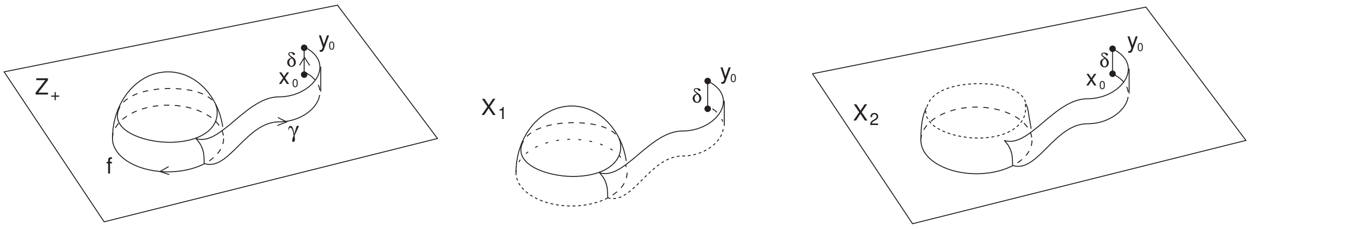
February 8, 2005

Some computations:

Gluing on a 2-disk: If X is a topological space and $f : \partial\mathbb{D}^2 \to X$ is continuous, then we can construct the quotient space $Z = (X \coprod \mathbb{D}^2)/\{x \sim f(x) : x \in \partial\mathbb{D}^2\}$, the result of gluing $\mathbb{D}^2$ to X along f. We can use Seifert - van Kampen to compute π_1 of the resulting space, although if we wish to be careful with basepoints x_0 (e.g., the image of f might not contain x_0, and/or we may wish to glue several disks on, in remote parts of X), we should also include a rectangle R, the mapping cylinder of a path γ running from $f(1,0)$ to x_0, glued to $\mathbb{D}^2$ along the arc from $(1/2,0)$ to $(1,0)$ (see figure). This space Z_+ deformation retracts to Z, but it is technically simpler to do our calculations with the basepoint y_0 lying above x_0. If we write $D_1 = \{x \in \mathbb{D}^2 : ||x|| < 1\} \cup (R \setminus X)$ and $D_2 = \{x \in \mathbb{D}^2 : ||x|| > 1/3\} \cup R$, then we can write $Z_+ = D_+ \cup (X \cup D_2) = X_1 \cup X_2$. But since $X_1 \simeq *$, $X_2 \simeq X$ (it is essentially the mapping cylinder of the maps f and γ) and $X_1 \cap X_2 = \{x \in \mathbb{D}^2 : 1/3 < ||x|| < 1\} \cap (R \setminus X) \sim S^1$, we find that

$$\pi_1(Z, y_0) \cong \pi_1(X_2, y_0) *_{\mathbb{Z}} \{1\} = \pi_1(X_2)/ < \mathbb{Z} >^N \cong \pi_1(X_2)/ < [\overline{\delta} * \overline{\gamma} * f * \gamma * \delta] >^N$$

If we then use δ as a path for a change of basepoint isomorphism, and then a homotopy equivalence from X_2 to X (fixing x_0), we have, in terms of group presentations, if $\pi_1(X, x_0) =< \Sigma | R >$, then $\pi_1(Z) =< \Sigma | R \cup \{[\overline{\gamma} * f * \gamma]\} >$. So the effect of gluing on a 2-disk on the fundamental group is to add a new relator, namely the word represented by the attaching map (adjusting for basepoint). All of this applies equally well to attaching several 2-disks; each adds a new relator.



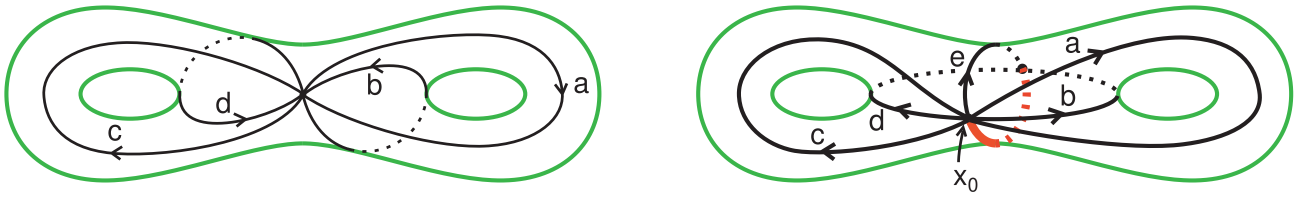
The inherent complications above derived from needing open sets can be legislated away, by introducing additional hypotheses:

Theorem: If $X = X_1 \cup X_2$ is a union of closed sets X_1, X_2, with $A = X_1 \cap X_2$ path-connected, and if X_1, X_2 have open neighborhood $\mathcal{U}_1, \mathcal{U}_2$ so that $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_1 \cap \mathcal{U}_2$ deformation retract onto X_1, X_2, A respectively, then $\pi_1(X) \cong \pi_1(X_1) *_{\pi_1(A)} \pi_1(X_2)$ as before.

The hypotheses are satisfied, for example, if $X_1.X_2$ are subcomplexes of the cell complex X.

This in turn opens up huge possibilities for the computation of $\pi_1(X)$. For example, for cell complexes, we can inductively compute π_1 by starting with the 1-skeleton, with free fundamental group, and attaching the 2-cells one by one, which each add a relator to the presentation of $\pi_1(X)$. [**Exercise:** (Hatcher, p.53, # 6) Attaching n-cells, for $n \geq 3$, has no effect on π_1.] For example, the 2-sphere S^2 can be thought of as a 2-disk with a 2-disk attached, along a circle, and so has $\pi_1(S^2) \cong \{1\}_{\mathbb{Z}}\{1\} = \{1\}$. We can also compute the fundamental group of any compact surface:

The *real projective plane* $\mathbb{R}P^2$ is the quotient of the 2-sphere S^2 by the antipodal map $x \mapsto -x$; it can also be thought of as the upper hemisphere, with identification only along the boundary. This in turn can be interpreted as a 2-disk glued to a circle, whose boundary wraps around the circle twice. So $\pi_1(\mathbb{R}P^2) \cong < a | a^2 > \cong \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. A surface F of genus 2 can be given a cell structure with 1 0-cell, 4 1-cells, and 1 2-cell, as in the figure, as in the first of the figures below. The fundamental group of the 1-skeleton is therefore free of rank 4, and $\pi_1(F)$ has a presentation with 4 generators and 1 relator. Reading the attaching map from the figure, the presentation is $< a, b, c, d \mid [a, b][c, d] >$.



Giving it a different cell structure, as in the second figure, with 2 0-cells, 6 1-cells, and 2 2-cells, after choosing a maximal tree, we can read off the two relators from the 2-cells to arrive at a different presentation $\pi_1(F) =< a, b, c, d, e \mid aba^{-1}eb^{-1}, cde^{-1}c^{-1}d^{-1} >$. A posteriori, these two presentations describe isomorphic groups.

Using the same technology, we can also see that, in general, any group is the fundamental group of some 2-complex X; starting with a presentation $G =< \Sigma | R >$, build X by starting with a bouquet of $|\Sigma|$ circles, and attach $|R|$ 2-disks along loops which represent each of the generators of R. (This works just as well for infinite sets Σ and/or R; essentially the same proofs as above apply.)