Math 971 Algebraic Topology

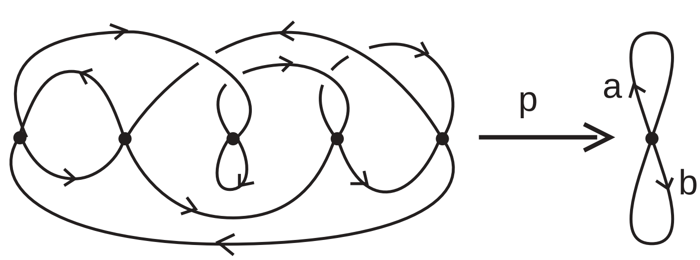
February 10, 2005

Postscript: why should we care? The role of the fundamental group in distinguishing spaces has already been touched upon; if two (path-connected) spaces have non-isomorphic fundamental groups, then the spaces are not homeomorphic, and even not homotopy equivalent. It is one of the most basic, and in many cases the best such invariant we have in our arsenal (hence the name "fundamental"). As we have seen with the circle, it captures the notion of how many times a loop "winds around" in a space. And the idea of using paths to understand a space is very basic; we explore a space by mapping familiar objects into it. (This is a theme we keep returning to in this course.) The concepts we have introduced play a role in analysis, for instance with the notion of a path integral; the invariance of the integral under homotopies rel endpoints is an important property, related to Green's Theorem and (locally) conservative vector fields. And the space of all paths in X plays an important (theoretical, although pprobably not practical) role in what we will do next.

Covering spaces: We can motivate our next topic by looking more closely at one of our examples above. The projective plane $\mathbb{R}P^2$ has $\pi_1 = \mathbb{Z}_2$. It is also the quotient of the simply-connected space S^2 by the antipodal map, which, together with the identity map, forms a group of homeomorphisms of S^2 which is isomorphic to $\mathbb{Z}_2$. The fact that $\mathbb{Z}_2$ has this dual role to play in describing $\mathbb{R}P^2$ is no accident; codifying this relationship requires the notion of a covering space.

The quotient map $q : S^2 \to \mathbb{R}P^2$ is an example of a *covering map*. A map $p : E \to B$ is called a covering map if for every point $x \in B$, there is a neighborhood $\mathcal{U}$ of x (an *evenly covered neighborhood*) so that $p^{-1}(\mathcal{U})$ is a disjoint union $\mathcal{U}_\alpha$ of open sets in E, each mapped homeomorphically onto $\mathcal{U}$ by (the restriction of) p . B is called the *base space* of the covering; E is called the *total space.* The quotient map q is an example; (the image of) the complement of a great circle in S^2 will be an evenly covered neighborhood of any point it contains. The disjoint union of 43 copies of a space, each mapping homeomorphically to a single copy, is an example of a *trivial covering.* As a last example, we have the famous exponential map $p : \mathbb{R} \to S^1$ given by $t \mapsto e^{2\pi i t} = (\cos(2\pi t), \sin(2\pi t))$. The image of any interval (a, b) of length less than 1 will have inverse image the disjoint union of the intervals $(a + n, b + n)$ for $n \in \mathbb{Z}$.

OK, maybe not the last. We can build many finite-sheeted (every point inverse is finite) coverings of a bouquet of two circles, say, by assembling n points over the vertex, and then, on either side, connecting the points by n (oriented) arcs, one each going in and out of each vertex. By choosing orientations on each 1-cell of the bouquet, we can build a covering map by sending the vertices above to the vertex, and the arcs to the one cells, homeomorphically, respecting the orientations. We can build infinite-sheeted coverings in much the same way.



Covering spaces of a (suitably nice) space X have a very close relationship to $\pi_1(X, x_0)$. The basis for this relationship is the

Homotopy Lifting Property: If $p : \widetilde{X} \to X$ is a covering map, $H : Y \times I \to X$ is a homotopy, $H(y, 0) = f(y)$, and $\widetilde{f} : Y \to \widetilde{X}$ is a *lift* of f (i.e., $p \circ \widetilde{f} = f$), then there is a unique lift $\widetilde{H}$ of H with $\widetilde{H}(y, 0) = \widetilde{f}(y)$.

The **proof** of this we will defer to next time, to give us sufficient time to ensure we finish it!

In particular, applying this property in the case $Y = \{*\}$, where a homotopy $H : \{*\} \times I \to X$ is really just a a path $\gamma : I \to X$, we have the **Path Lifting Property**: "given a covering map $p : \widetilde{X} \to X$, a path $\gamma : I \to X$ with $\gamma(0) = x_0$, and a point $\widetilde{x}_0 \in p^{-1}(x_0)$, there is a unique path $\widetilde{\gamma}$ lifting γ with $\widetilde{\gamma}(0) = \widetilde{x}_0$." One of the immediate consequences of this is one of the cornerstones of covering space theory:

If $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ is a covering map, then the induced homomorphism $p_* : \pi_1(\widetilde{X}, \widetilde{x}_0) \to \pi_1(X, x_0)$ is injective.

Proof: Suppose $\gamma : (I, \partial I) \to (\widetilde{X}, \widetilde{x}_0)$ is a loop $p_*([\gamma]) = 1$ in $\pi_1(X, x_0)$. So there is a homotopy $H : (I \times I, \partial I \times I) \to (X, x_0)$ between $p \circ \gamma$ and the constant path. By homotopy lifting, there is a homotopy $\widetilde{H}$ from γ to the lift of the constant map at x_0. The vertical sides $s \mapsto \widetilde{H}(0, s), \widetilde{H}(1, s)$ are also lifts of the constant map, beginning from $\widetilde{H}(0, 0), \widetilde{H}(1, 0) = \gamma(0) = \gamma(1) = \widetilde{x}_0$, so are the constant map at $\widetilde{x}_0$. Consequently, the lift at the bottom is the constant map at $\widetilde{x}_0$. So $\widetilde{H}$ represents a null-homotopy of γ, so $[\gamma] = 1$ in $\pi_1(\widetilde{X}, \widetilde{x}_0)$. So $\pi_1(\widetilde{X}, \widetilde{x}_0) = \{1\}$.

Even more, the image $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0))) \subseteq \pi_1(X, x_0)$ is precisely the elements whose representatives are loops at x_0, which when lifted to paths starting at $\widetilde{x}_0$), are loops. For if γ lifts to a loop $\widetilde{\gamma}$, then $p \circ \widetilde{\gamma} = \gamma$, so $p_*([\widetilde{\gamma}]) = [\gamma]$. Conversely, if $p_*([\widetilde{\gamma}]) = [\gamma]$, then γ and $p \circ \widetilde{\gamma}$ are homotopic rel endpoints, and so the homotopy lifts to a homotopy rel endpoints between the lift of γ at $\widetilde{x}_0$, and the lift of $p \circ \widetilde{\gamma}$ at $\widetilde{x}_0$ (which is $\widetilde{\gamma}$, since $\widetilde{\gamma}(0) = \widetilde{x}_0$ and lifts are unique). So the lift of γ is a loop, as desired.