Math 971 Algebraic Topology

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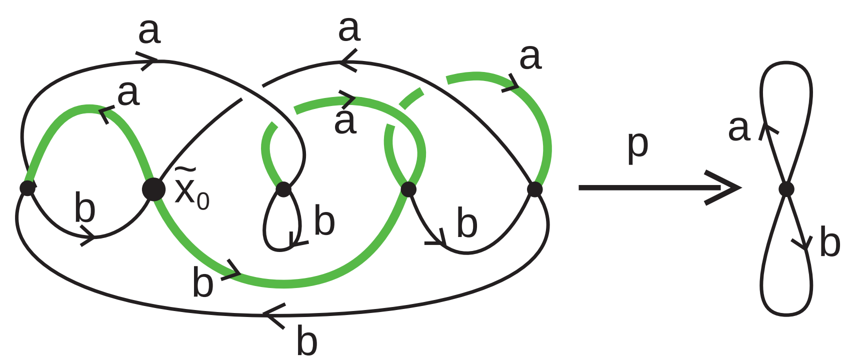
The **proof** of the homotopy lifting property follows a pattern that we will become very familiar with: we lift maps a little bit at a time. For every $x \in X$ there is an open set $\mathcal{U}_x$ evenly covered by p . For each fixed $y \in Y$, since I is compact and the sets $H^{-1}(\mathcal{U}_x)$ form an open cover of $Y \times I$, then since I is compact, the Tube Lemma provides an open neighborhood $\mathcal{V}$ of y in Y and finitely many $p^{-1}\mathcal{U}_x$ whose union covers $\mathcal{V} \times I$.

To define $\widetilde{H}(y,t)$, we (using a Lebesgue number argument) cut the interval $\{y\} \times I$ into finitely many pieces, the ith mapping into $\mathcal{U}_{x_i}$ under H. $\widetilde{f}(y)$ is in one of the evenly covered sets $\mathcal{U}_{x_1\alpha_1}$, and the restricted map $p^{-1} : \mathcal{U}_{x_1} \to \mathcal{U}_{x_1\alpha_1}$ following H restricted to the first interval lifts H along the first interval to a map we will call $\widetilde{H}$. We then have lifted H at the end of the first interval = the beginning of the second, and we continue as before. In this way we can define $\widetilde{H}$ for all (y,t) . To show that this is independent of the choices we have made along the way, we imagine two ways of cutting up the interval $\{y\} \times I$ using evenly covered neighborhoods $\mathcal{U}_{x_i}$ and $\mathcal{V}_{w_j}$, and take intersections of both sets of intervals to get a common refinement of both sets, covered by the intersections $\mathcal{U}_{x_i} \cap \mathcal{V}_{w_j}$, and imagine building $\widetilde{H}$ using the refinement. At the start, at $\widetilde{f}(y)$, we are in $\mathcal{U}_{x_1\alpha_1} \cap \mathcal{V}_{w_1\beta_1}$. Because at the start of the lift $(y,0)$ we lift to the same point, and p^{-1} restricted to this intersection agrees with p^{-1} restricted to each of the two pieces, we get the same lift acroos the first refined subinterval. This process repeats itself across all of the subintervals, showing that the lift is independent of the choices made. This also shows that the lift is unique; once we have decided what $\widetilde{H}(y,0)$, the rest of the values of the $\widetilde{H}$ are determined by the requirement of being a lift. also, once we know the map is well-defined, we can see that it is continuous, since for any y, we can make the same choices across the entire open set V given by the Tube Lemma, and find that $\widetilde{H}$, restricted to $\mathcal{V} \times (a_i - \delta, b_i + \delta)$ (for a small delta; we could wiggle the endpoints in the construction without changing the resulting function, by its well-definedness) is H estricted to this set followed by p^{-1} restriced in domain and range, so this composition is continuous. So $\widetilde{H}$ is locally continuous, hence continuous.

So, for example, if we build a 5-sheeted cover of the bouquet of 2 circles, as before, (after choosing a maximal tree upstairs) we can read off the images of the generators of the fundamental group of the total space; we have labelled each ede by the ereator it traces out downstairs, and for each ede outside of the maximal tree chosen, we read from basepoint out the tree to one end, across the edge, and then back to the basepoint in the tree. In our example, this gives:



$$< ab, aaab^{-1}, baba^{-1}, baa, ba^{-1}bab^{-1}, bba^{-1}b^{-1} | >$$

This is (from its construction) a copy of the free group on 6 letters, in the free group $F(a,b)$. In a similar way, by explicitly building a covering space, we find that the fundamental group of a closed surface of genus 3 is a subgroup of the fundamental group of the closed surface of genus 2.

The cardinality of a point inverse $p^{-1}(y)$ is, by the evenly covered property, constant on (small) open sets, so the set of points of x whose point inverses have any given cardinality is open. Consequently, if X is connected, this number is constant over all of X, and is called the number of *sheets* of the covering $p : \widetilde{X} \to X$.