Math 971 Algebraic Topology

April 5, 2005

A "Seifert - van Kampen" theorem for homology? Start with $X = A \cup B$; try to express the homology of X in terms of that of A, B, and $A \cap B$. Using LESs, we might try to first build an SES out of the chain complexes $C_*(A \cap B), C_*(A), C_*(B)$, and $C_*(X)$. Taking our cue from the proof of S-vK, think of chains in X as sums of chains in A and B, modding out by chains in $A \cap B$. So we try:

$$0 \to C_n(A \cap B) \to C_n(A) \oplus C_n(B) \to C_n(X) \to 0$$

where $j_n : C_n(A) \oplus C_n(B) \to C_n(X)$ is defined as $j_n(a, b) = a + b$. In order to get exactness at the middle term, we set $i_n : C_n(A \cap B) \to C_n(A) \oplus C_n(B)$ to be $i_n(x) = (x, -x)$, since $C_n(A \cap B) = C_n(A) \cap C_n(B)$. i_n is then injective, and this sequence is exact at the middle term. But, in general, j_n is far from surjective. But we can replace $C_n(X)$ with the image of j_n, calling it $C_n^{\{A,B\}}(X)$; these are chains where simplices map into either Aor B. Then we get a SES, and hence a LES in homology. This uses a "new" homology group $H_n^{\{A,B\}}(X)$. But, like S-vK, under the right conditions, $H_n^{\{A,B\}}(X) \cong H_n(X)$!

Starting from scratch, the idea is that, starting with an *open cover* $\{\mathcal{U}_\alpha\}$ of X (or, more generally, with a collections of subspaces A_α whose interiors $\mathcal{U}_\alpha$ cover X), we build the *chain groups subordinate to the cover* $C_n^{\mathcal{U}}(X) = \{\sum a_i \sigma_i^n : \sigma_i : n : \Delta^n \to X, \sigma_i^n(\Delta^n) \subseteq \mathcal{U}_\alpha \text{ for some } \alpha\} \subseteq C_n(X)$. Since the face of any simplex mapping into $\mathcal{U}_\alpha$ also maps into $\mathcal{U}_\alpha$, our ordinary bdry maps induce bdry maps on these groups, turning $(C_n^{\mathcal{U}}(X), \partial_n)$ into a chain complex. Our main result is that the inclusion i of these groups into $C_n(X)$ induces an iso on homology. And to show this, we use the SES of chain complexes

$$0 \to C_n^{\mathcal{U}}(X) \xrightarrow{i} C_n(X) \to C_n(X)/C_n^{\mathcal{U}}(X) \to 0$$

to build a LES. Every third group is $H_n(C_*(X)/C_*^{\mathcal{U}}(X))$; we show these groups are 0, so i_* will be an isomorphism. Working back through the definition of $H_n(C_*(X)/C_*^{\mathcal{U}}(X))$, we need to show that if $z \in C_n(X)$ with $\partial z \in C_{n-1}^{\mathcal{U}}(X)$ (i.e., z is a relative cycle), then there is a $w \in C_{n+1}(X)$ with $z - \partial w \in C_n^{\mathcal{U}}(X)$ (i.e., z is a relative bdry). In words, if z has bdry a sum of "small" simplices, then there is a chain z' made of small simplices so that $z - z'$ is a bdry.

And the key to building z' and w is a process known as *barycentric subdivision*. The idea is to cut an n-simplex into smaller n-simplices, in a way compatible with the boundary map.The *barycenter* of a simple $[v_0, \dots v_n]$ is the point $(v_0 + \cdots + v_n)/((n+1)$. Playing with 1- and 2-simplices, we are led to the idea that we cut an n-simplex into $(n+1)!$ simplices; each new simplex is the convex span of vertices chosen as (vertex) , (barycenter of a 1-simplex having (vertex) as a vertex), (barycenter of a 2-simplex containing the previous 2 vertices), etc.. Taking into account orientations as well, we define the barycentric subdivision of a singular n-simplex $\sigma : [v_0, \dots , v_n] \to X$ to be

$$S(\sigma) = \sum_\alpha \text{sgn}(\alpha)\sigma|_{[v_{\alpha(0)},(v_{\alpha(0)}+v_{\alpha(1)})/2,(v_{\alpha(0)}+v_{\alpha(1)}+v_{\alpha(2)})/3,\dots,(v_{\alpha(0)}+\cdots v_{\alpha(n)})/(n+1)]}$$

where the sum is taken over all permutations of $\{0, \dots , n\}$. This gives the subdivision operator, $S : C_n(X) \to C_n(X)$. A "routine" calculation establishes that $\partial S = S\partial$, i.e., it is a chain map. All of the subsimplices in the sum are a definite factor smaller than the original simplex; if the diameter of $[v_0, \dots , v_n]$ is d, then every individual simplex in $S(\sigma)$ will have diameter at most $nd/(n + 1)$. So by repeatedly applying the subdivision operator S to a singular simplex, we will obtain a singular chain $S^k(\sigma)$, which is "really" σ written as a sum of tiny simplices, whose singular simplices have image as small as we want. Or put more succinctly, if $\{\mathcal{U}_\alpha\}$ is an open cover of X and $\sigma : \Delta^n \to X$ is a singular n-simplex, then choosing a Lebesgue number ϵ for the open cover $\sigma^{-1}(\mathcal{U}_\alpha)$ of the compact metric space Δ^n, and choosing a k with $d(n/(n+1))^k < \epsilon$, we find that $S^k(\sigma)$ is a sum of singular simplices each of which maps into one of the $\mathcal{U}_\alpha$, i.e., $S^k(\sigma) \in C_n^{\mathcal{U}}(X)$.

In the end, we will choose our needed "small" cycle to be $z' = S^k z$. And to show that their difference is a boundary, we will build a chain homotopy between Id and S^k. And to do that, we define a map $R : C_n(X) \to C_{n+1}(X \times I)$; when followed by the projection-induced map $p_\# : C_{n+1}(X \times I) \to C_{n+1}(X)$, we get a map $T : C_n(X) \to C_{n+1}(X)$, and we show that $\partial T + T\partial = I - S$. Then we set $H = \sum TS^j$, where the sum is taken over $j = 0, \dots k - 1$. Then we have $\partial H_k + H_k\partial = \sum \partial TS^j + TS^j\partial = \sum(\partial T + T\partial)S^j = \sum(S^j - S^{j+1}) = I - S^k$ (since the last sum telescopes). And defining R, is, formally, just another particular sum. Setting up some notation, thinking of $\Delta^n \times I$, as before, as having vertices $\{v_0, \dots v_n\}$ on the 0-end and $\{w_0, \dots , w_n\}$ on the 1-end, $N = \{0, \dots , n\}$, $\Pi(Q)$ = the group of permutations of Q, and $\sigma' = \sigma \times I : \Delta^n \times I \to X \times I$), we have

$$R(\sigma) = \sum_{A \subseteq N} \sum_{\pi \in \Pi(N \setminus A)} \{(-1)^{|A|}\text{sgn}(\pi) \prod_{j \in N \setminus A} (-1)^j\} \sigma'|_{[v_{i_0},\dots,v_{i_j},(w_{i_0}+\cdots w_{i_j})/(j+1),(w_{i_0}+\cdots w_{i_j}+w_{\pi(i_{j+1})})/(j+2),\dots,(w_{i_0}+\cdots w_{i_j}+w_{\pi(i_{j+1})}+\cdots w_{\pi(i_n)})/(n+1)]}$$

where we sum over all non-empty subsets of $\{0, \dots n\}$ (with the induced ordering on vertices from the ordering on $\{0, \dots , n\}$). Intuitively, this map "interpolates" between the simplex $[v_0, \dots v_n]$ and the barycentric subdivision on $w_0, \dots , w_n$, by taking the (signed sums of the) convex spans of simplices on the bottom (0) and simplices on the top (1). Again, a "routine" calculation will establish that $\partial T + T\partial = I - S$, as desired. [At any rate, I verified it for n=1,2; the formula for the sign of each simplex was determined by working backwards from these examples.]