Math 971 Algebraic Topology

April 12, 2005

There is another piece of homological algebra that we will find useful ; the Five Lemma. It allows us to compare the information contained in two LESs.

Five Lemma: If we have abelian groups and maps

$$\begin{array}{ccccccccc} A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \xrightarrow{h_n} & D_n & \xrightarrow{i_n} & E_n \\ \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \delta\downarrow & & \epsilon\downarrow \\ A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \xrightarrow{h_{n-1}} & D_{n-1} & \xrightarrow{i_{n-1}} & E_{n-1} \end{array}$$

where the rows are exact, the squares commute, and the maps $\alpha,\beta,\delta,\epsilon$ are all isomorphisms, then γ is an isomorphism.

To show injectivity, suppose $x\in C_n$ and $\gamma x=0$, then $h_{n-1}\gamma x=\delta h_n x=0$, so, since δ is injective, $h_n x=0$. So by the exactness at C_n, $x=g_n y$ for some $y\in B_n$. Then $g_{n-1}\beta y=\gamma g_n y=\gamma x=0$, so by exactness at B_{n-1}, $\beta y=f_{n-1}z$ for some $z\in A_{n-1}$. Then since α is surjective, $f_{n-1}z=\alpha w$ for some w. Then $0=g_n f_n w$. But $\beta f_n w=f_{n-1}\alpha w$ $f_{n-1}z=\beta y$, so since β is injective, $y=f_n w$. So $0=g_n f_n w=g_n y=x$. So $x=0$. For surjectivity, suppose $x\in C_{n-1}$. Then $h_{n-1}x\in D_{n-1}$, so since δ is surjective, $h_{n-1}x=\delta y$ for some $y\in D_n$. Then $\epsilon i_n y=i_{n-1}\delta y=i_{n-1}h_{n-1}x=0$, so since ϵ is injective, $i_n y=0$. So by exactness at D_n, $y=h_n z$ for some $z\in C_n$. Then $h_{n-1}\gamma z=\delta h_n z=\delta y=h_{n-1}x$, so $h_{n-1}(\gamma z-x)=0$, so by exactness at C_{n-1}, $\gamma z-x=g_{n-1}w$ for some $w\in B_{n-1}$. Then since β is surjective, $w=\beta u$ for some $u\in B_n$. Then $\gamma g_n u=g_{n-1}\beta u=g_{n-1}w=\gamma z-x$, so $x=\gamma z-\gamma g_n u=\gamma(z-g_n u)$.

The second result that this machinery gives us is what is properly known as *excision*: If $B\subseteq A\subseteq X$ and $\mathrm{cl}_X(B)\subseteq \mathrm{int}_X(A)$, then for every k the inclusion-induced map $H_k(X\setminus B,A\setminus B)\to H_k(X,A)$ is an isomorphism.

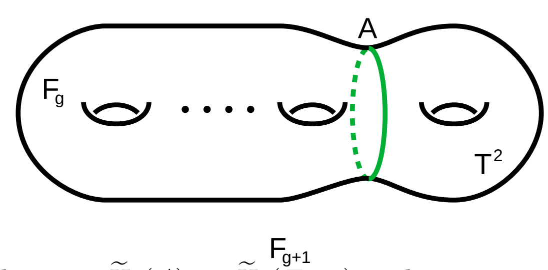
An equivalent formulation of this is that if $A,B\subseteq X$ and $\mathrm{int}_X(A)\cup\ \mathrm{int}_X(B)=X$, then the inclusion-induced map $H_k(B,A\cap B)\to H_k(X,A)$ is an isomorphism. [Set $B'=X\setminus B$.] To prove the second statement, we know that the inclusion $C_n^{\{A,B\}}(X)\to C_n(X)$ induces isomorphisms on homology, as does $C_n(A)\to C_n(A)$, so, by the five lemma, the induced map $C_n^{\{A,B\}}(X)/C_n(A)\to C_n(X)/C_n(A)=C_n(X,A)$ induces isomorphisms on homology. But the inclusion $C_n(B)\to C_n^{\{A,B\}}(X)$ induces a map $C_n(B,A\cap B)=C_n(B)/C_n(A\cap B)\to C_n^{\{A,B\}}(X)/C_n(A)$ which is an isomorphism of chain groups; a basis for $C_n^{\{A,B\}}(X)/C_n(A)$ consists of singular simplices which map into A or B, but don't map into A, i.e., of simplices mapping into B but not A, i.e., of simplices mapping into B but not $A\cap B$. But this is the same as the basis for $C_n(B,A\cap B)$!

With these tools, we can start making some real homology computations. First, we show that if $\emptyset\neq A\subseteq X$ is "nice enough", then $H_n(X,A)\cong\widetilde{H}_n(X/A)$. The definition of nice enough, like Seifert - van Kampen, is that A is closed and has an open nbhd $\mathcal{U}$ that deformation retracts to A (think: A is the subcomplex of a CW-complex X). Then using $\mathcal{U},X\setminus A$ as a cover of X, and $\mathcal{U}/A,(X\setminus A)/A$ as a cover of X/A, we have

$$\widetilde{H}_n(X/A)\overset{(1)}{\cong}H_n(X/A,A/A)\overset{(2)}{\cong}H_n(X/A,\mathcal{U}/A)\overset{(3)}{\cong}H_n(X/A\setminus A/A,\mathcal{U}/A\setminus A/A)\overset{(4)}{\cong}H_n(X\setminus A,\mathcal{U}\setminus A)\overset{(5)}{\cong}H_n(X,A)$$

where (1),(2) follow from the LES for a pair, (3),(5) by excision, and (4) because the restriction of the map $X\to X/A$ gives a homeo of pairs.

Second, if X,Y are T_1, $x\in X$ and $y\in Y$ each have neighborhoods $\mathcal{U},\mathcal{V}$ which deformation retract to each point, then the one-point union $Z=X\vee Y=(X\coprod Y)/(x=y)$ has $\widetilde{H}_n(Z)\cong\widetilde{H}_n(X)\oplus\widetilde{H}_n(Y)$; this follows from a similar sequence of isomorphisms. Setting $z=$ the image of $\{x,y\}$ in Z, we have $\widetilde{H}_n(Z)\cong H_n(Z,z)\cong H_n(Z,\mathcal{U}\vee\mathcal{V})\cong H_n(Z\setminus z,\mathcal{U}\vee\mathcal{V}\setminus z)\cong H_n([X\setminus x]\coprod[Y\setminus y],[\mathcal{U}\setminus x]\coprod[\mathcal{V}\setminus y])\cong H_n(X\setminus x,\mathcal{U}\setminus x)\oplus H_n(Y\setminus y,\mathcal{V}\setminus y)\cong H_n(X,x)\oplus H_n(Y,y)\cong\widetilde{H}_n(X)\oplus\widetilde{H}_n(Y)$. By induction, we then have $\widetilde{H}_n(\vee_{i=1}^k X_i)\cong\oplus_{i=1}^k\widetilde{H}_n(X_i)$



As an application, we can compute the homology groups of the closed orientable surfaces of genus g, F_g. We will argue by induction on g that $\widetilde{H}_i(F_g)=\mathbb{Z}$ for $i=2$, $\mathbb{Z}^{2g}$ for $i=1$, and 0 for all other i. For the base case g=1 we have the 2-torus, whose homology we have computed previously. For the inductive step, we look at the homology of the pair (F_{g+1},A) depicted here; with what we have learned above, in the LES for the pair, we will have $H_i(F_{g+1},A)\cong\widetilde{H}_i(F_{g+1}/A)\cong\widetilde{H}_i(F_g\vee T^2)\cong\widetilde{H}_i(F_g)\oplus\widetilde{H}_i(T^2)$, which we can compute, by our inductive hypothesis. And so we find that we have $\widetilde{H}_2(A)\to\widetilde{H}_2(F_{g+1})\to\widetilde{H}_2(F_g\oplus\widetilde{H}_2(T^2)\to\widetilde{H}_1(A)\to\widetilde{H}_1(F_{g+1})\to\widetilde{H}_1(F_g)\oplus\widetilde{H}_2(T^2)\to\widetilde{H}_0(A)$

which renders as $0\to\widetilde{H}_2(F_{g+1})\to\mathbb{Z}^2\to\mathbb{Z}\to\widetilde{H}_1(F_{g+1})\to\mathbb{Z}^{2g+2}\to 0$. The point, though is that the map $\widetilde{H}_1(A)\to\widetilde{H}_1(F_{g+1})$ is the 0 map, since the circle A is null-homologous in F_{g+1}; it bounds the once-punctured torus X on the right of the figure. [Formally, what we mean is that the generator of $\widetilde{H}_1(A)$, a singular 1-simplex wrapping once around A, is a boundary in $C_1(F_{g+1})$; writing X as a sum of singular 2-simplices provides a demonstration.] So we can cut our exact sequence into two pieces $0\to\widetilde{H}_2(F_{g+1})\to\mathbb{Z}^2\to\mathbb{Z}\to 0$ and $0\to\widetilde{H}_1(F_{g+1})\to\mathbb{Z}^{2g+2}\to 0$ The first implies that $\widetilde{H}_2(F_{g+1})$ is a direct summand, with $\mathbb{Z}$, for $\mathbb{Z}^2$, and so is $\mathbb{Z}$; the second asserts that $\widetilde{H}_1(F_{g+1})\cong\mathbb{Z}^{2g+2}$, as desired. So our inductive step is proved, and our computation follows by induction.