Math 971 Algebraic Topology

April 26, 2005

The Hurewicz map $H : \pi_1(X) \to H_1(X)$ induces, when X is path-connected, an isomorphism from $\pi_1(X)/[\pi_1(X), \pi_1(X)]$ to $H_1(X)$. This result can be used in two ways; knowing a (presentation for) $\pi_1(X)$ allows us to compute $H_1(X)$, by writing the relators additively, giving $H_1(X)$ as the free abelian group on the generators, modulo the kernel of the "presentation matrix" given by the resulting linear equations. Conversely, knowing $H_1(X)$ provides information about $\pi_1(X)$. For example, a calculation on the way to invariance of domain implied that for every knot K in S^3 (i.e., the image of an embedding $h : S^1 \hookrightarrow S^3$), $H_1(S^3 \setminus K) \cong bbz$. This implies that the abelianization of $G_K = \pi_1(S^3 \setminus K)$ (i.e., the largest abelian quotient of G_K is $\mathbb{Z}$. But this in turn implies that for every integer $n \geq 2$, there is a unique surjective homomorphism $G_K \to \mathbb{Z}_n$, since such a homomorphism must factor through the abelianization, and there is exactly one surjective homomorphism $\mathbb{Z} \to \mathbb{Z}_n$! Consequently, there is a unique (normal) subroup (the kernel of this homomorphism) $K_n \subseteq G_K$ with quotient $\mathbb{Z}_n$. Using the Galois correspondence, there is a (unique) covering space X_n of $X = S^3 \setminus K$ corresponding to K_n, called the n-fold cyclic covering of K . This space is determined by K and n, and so its homology groups are determined by the same data. And even though homology cannot distinguish between two knot complements, K, K', it might be the case that homology can distinguish between their cyclic coverings. Consequently, if $H_1(X_n) \not\cong H_1(X'_n)$, then K and K' have non-homeomorphic complement, and so represent "different" embeddings, hence different knots. In practice, one can compute presentations for $\pi_1(X_n)$ (in several different ways), and so one can compute $H_1(X_n)$, providing an effective way to use homology to distinguish knots! This approach was ultimately formalized (by Alexander) into a polynomial invariant of knots, known as the Alexander polynomial.

Computing the homology of the cyclic coverings can be done in several ways. The Reidemeister-Schreier method will allow one to compute a presentation for the kernel of a homomorphism $\varphi : G \to H$, given a presentation of G and a *transversal* of the map, which is a representative of each coset of G modulo the kernel. Abelianizing this will give homology computation. Another approach uses *Seifert surfaces*, orientable surfaces with $\partial\Sigma = K$, to cut $S^3 \setminus K$ open along. Writing $S^3 \setminus K = (S^3 \setminus N(\Sigma)) \cup N(\Sigma)$ allows us to use Mayer-Vietoris to compute homology. But the cyclic covering spaces can be built by "unwinding" this view of $S^3 \setminus K$; instead of gluing the two ends of $N(K)$ to the same $S^3 \setminus N(\Sigma)$, we can take n copies of $S^3 \setminus N(\Sigma)$ and glue them together in a circle. Mayer-Vietoris again tells us how to compute the homology of the resulting space. Details may be found on the accompanying pages taken from Rolfsen's "Knots and Links".

2. EXERCISE : The proof given above shows how to obtain a Seifert surface from a map $X \to S^1$ defined on the complement of a link. Show that all Seifert surfaces arise in this way. I. e., given a Seifert surface M for L^n there exists a map $F: S^{n+2} - L^n \to S^1$ and a point x in S^1 such that $M = F^{-1}(x)$ and moreover F^{-1} of a neighbourhood of x is a bicollar on $\mathring{M}$. [Hint: send everything outside a given bicollar of $\mathring{M}$ to a point].

3. REMARK : Recent work in topological transversality enables the above existence theorem to go through for topological links, with suitable additional hypotheses. That it does not work in certain cases (in dimension four) is pointed out in recent work of Cappell and Shaneson.

C. CONSTRUCTION OF THE CYCLIC COVERINGS OF A KNOT COMPLEMENT USING SEIFERT SURFACES.

There is an important class of covering spaces of a knot complement $X = S^{n+2} - K^n$, which will be used in the next chapter to define certain abelian invariants of K. Readers unfamiliar with covering space theory will find a synopsis in Appendix A.

Seifert surfaces give a convenient means of constructing these covering spaces, in a manner entirely analogous to 'cuts' in the classical theory of Riemann surfaces.

Let M^{n+1} be a Seifert surface for the knot K^n in S^{n+2} and let $N : \mathring{M} \times (-1,1) \to S^{n+2}$ be an open bicollar of the interior $\mathring{M} = M - K$, so $\mathring{M} = N(\mathring{M} \times 0)$. We denote:

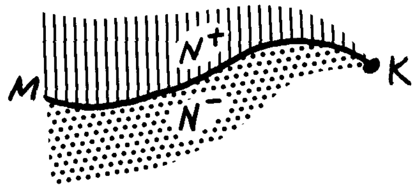
$N = N(\mathring{M} \times (-1,1))$

$N^+ = N(\mathring{M} \times (0,1))$

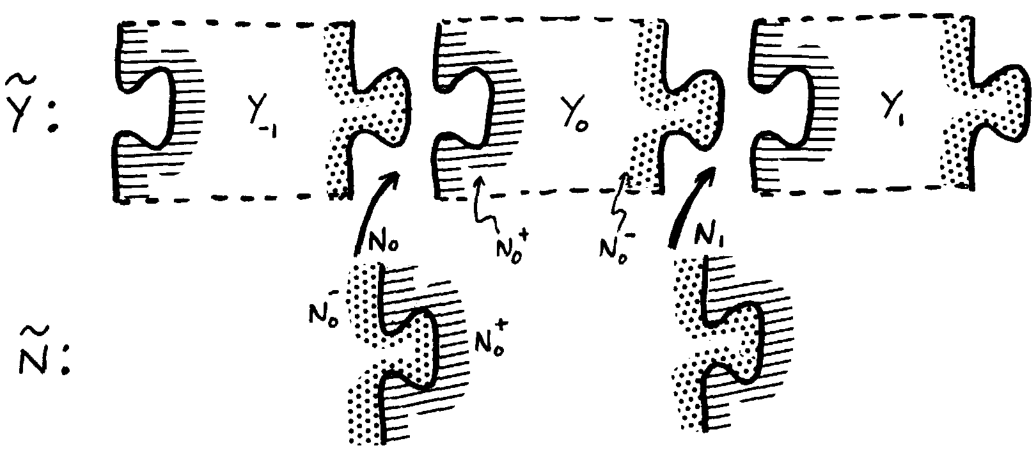
$N^- = N(\mathring{M} \times (-1,0))$

$Y = S^{n+2} - M$

$X = S^{n+2} - K$



Thus we have two triples (N,N^+,N^-) and (Y,N^+,N^-). Form countably many copies of each, denoted (N_i,N_i^+,N_i^-) and (Y_i,N_i^+,N_i^-), $i = 0, \pm 1, \pm 2, \cdots$. Let $\tilde{N} = \bigcup_{i=-\infty}^{\infty} N_i$ and $\tilde{Y} = \bigcup_{i=-\infty}^{\infty} Y_i$ be the disjoint unions. Finally, form an identification space by identifying $N_i^+ \subset Y_i$ with $N_i^+ \subset N_i$ via the identity homeomorphism, and likewise identify each $N_i^- \subset Y_i$ with $N_{i+1}^- \subset N_{i+1}$. Call the resulting space $\tilde{X}$.



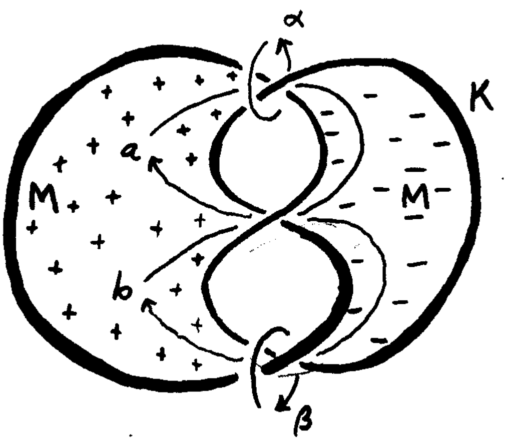
EXERCISE. Verify the following facts. $\tilde{X}$ is a path-connected open (n+2) - manifold. There is a map $p : \tilde{X} \to X$ which is a regular covering homology of cyclic covers.

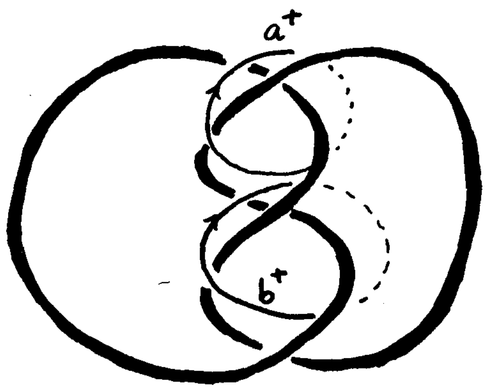
B. CALCULATION USING SEIFERT SURFACES.

An example will best illustrate the method.

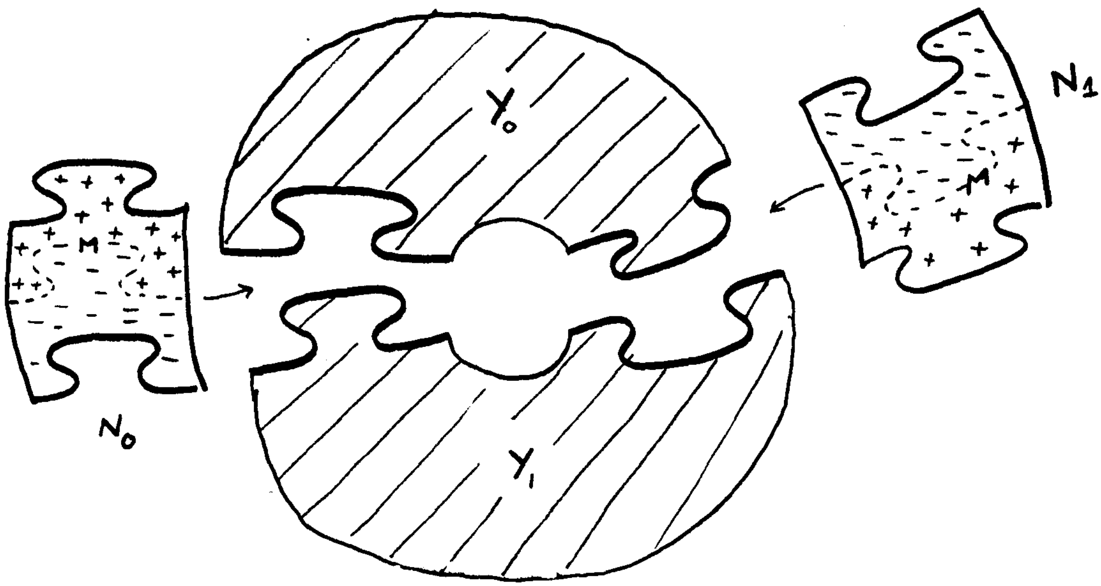
1. EXAMPLE : K = trefoil in S^3. Find the homology of the two-fold cyclic cover $\tilde{X}_2$ of $X = S^3 - K$.

Consider the Seifert surface M pictured





As in the previous chapter, we construct $\tilde{X}_2$ from two copies Y_o, Y_1 of $Y = S^3 - M$ and glue them together by N_o, N_1, each homeomorphic with $\mathring{M} \times (-1,1)$ according to the schematic



the twofold cyclic cover of $X = S^3 - K$

Now $H_1(\mathring{M})$ and $H_1(Y)$ are both free abelian with respective bases a, b and α, β as indicated in the first figure. Pushing a, b off $\mathring{M}$ and into N^+ or N^-, we see from the picture that, in $H_1(Y)$, the following equations hold :

$$a^- = \beta - \alpha \qquad a^+ = -\alpha$$

$$b^- = -\beta \qquad b^+ = \alpha - \beta$$

To compute $H_1(\tilde{X}_2)$ note that the subset $Y_o \cup Y_1$ has homology generators α_o, β_o, α_1, β_1 corresponding to α, β. Now putting in N_o introduces relations

$$\beta_1 - \alpha_1 = -\alpha_o \qquad (1) \quad (a_o^- = a_o^+)$$

$$-\beta_1 = \alpha_o - \beta_o \qquad (2) \quad (b_o^- = b_o^+)$$

and adding N_1 introduces relations

$$\beta_o - \alpha_o = -\alpha_1 \qquad (3) \quad (a_1^- = a_1^+)$$

$$-\beta_o = \alpha_1 - \beta_1 \qquad (4) \quad (b_1^- = b_1^+)$$

There is also a nontrivial 1-cycle γ which runs once around $\tilde{X}_2$ and we have the abelian group presentation :

$$H_1(\tilde{X}_2) \cong (\alpha_o, \beta_o, \alpha_1, \beta_1, \gamma; \text{ relations (1)-(4) })$$

Use (2) and (3) to eliminate α_1, β_1 :

$$H_1(\tilde{X}_2) \cong (\alpha_o, \beta_o, \gamma; \ \beta_o = 2\alpha_o, \ \alpha_o = 2\beta_o \ )$$

$$\cong (\alpha_o, \gamma; \ 3\alpha_o = 0 \ )$$

$$\cong Z \oplus Z/3 \ .$$

2. JUSTIFICATION OF THE CALCULATION.

It is convenient to connect the pieces by a curve $\Gamma \subset \tilde{X}_2$ which lies over a small loop in X linking K once. Let

$$Y' = Y_o \cup Y_1 \cup \Gamma$$

$$N' = N_o \cup N_1 \cup \Gamma$$

and observe that

$$Y' \cup N' = \tilde{X}_2$$

$$Y' \cap N' = N_o^+ \cup N_o^- \cup N_1^+ \cup N_1^- \cup \Gamma$$

Observing that $Y' \cap N'$ is connected, the exact Mayer-Vietoris sequence of reduced homology :

$$\ldots \longrightarrow H_1(N' \cap Y') \xrightarrow{\;f\;} H_1(N') \oplus H_1(Y') \longrightarrow H_1(\tilde{X}_2) \longrightarrow 0$$

shows that $H_1(\tilde{X}_2)$ is isomorphic with the cokernel of f. The other groups in the diagram are free abelian, with bases :

$H_1(N' \cap Y')$: $a_o^+, b_o^+, a_o^-, b_o^-, a_1^+, b_1^+, a_1^-, b_1^-, \gamma$

$H_1(N')$: $a_o, b_o, a_1, b_1, \gamma'$

$H_1(Y')$: $\alpha_o, \beta_o, \alpha_1, \beta_1, \gamma''$

where $\gamma, \gamma', \gamma''$ are all names for the 1-cycle carried by Γ. Now f is just the sum of the two inclusion-induced homomorphisms. In terms of the bases, f is the map :

$a_o^+ \longrightarrow (a_o, -\alpha_o)$	$a_1^+ \longrightarrow (a_1, -\alpha_1)$
$b_o^+ \longrightarrow (b_o, \alpha_o - \beta_o)$	$b_1^+ \longrightarrow (b_1, \alpha_1 - \beta_1)$
$a_o^- \longrightarrow (a_o, \beta_1 - \alpha_1)$	$a_1^- \longrightarrow (a_1, \beta_o - \alpha_o)$
$b_o^- \longrightarrow (b_o, -\beta_1)$	$b_1^- \longrightarrow (b_1, -\beta_o)$
	$\gamma \longrightarrow (\gamma', \gamma'')$

The reader should check that calculation of the free abelian group $H_1(N') \oplus H_1(Y')$ of rank 10, modulo the image of f, is equivalent to the calculation in the example above.

3. EXERCISE : Show that for this example $H_p(\tilde{X}_2) = 0, \quad p \geq 2.$

4. EXERCISE : Show that for the trefoil :

$H_1(\tilde{X}_3) \cong Z \oplus Z/2 \oplus Z/2$ $\qquad H_1(\tilde{X}_6) \cong Z \oplus Z \oplus Z$

$H_1(\tilde{X}_4) \cong Z \oplus Z/3$ $\qquad H_1(\tilde{X}_7) \cong Z$

$H_1(\tilde{X}_5) \cong Z$ $\qquad H_1(\tilde{X}_8) \cong Z \oplus Z/3$

5. EXERCISE : Show that for the figure-eight knot,*

$$H_1(\tilde{X}_2) \cong Z \oplus Z/5$$

$$H_1(\tilde{X}_3) \cong Z \oplus Z/4 \oplus Z/4$$

$$H_1(\tilde{X}_4) \cong Z \oplus Z/3 \oplus Z/15$$

6. EXERCISE : Show in general that $H_1(\tilde{X}_k)$ has a Z summand.

7. EXERCISE : Recall the definition of connected sum of knots in S^3 (section 2G). Show that if $K \cong K' \# K''$ and $\tilde{X}_k$, $\tilde{X}'_k$ and $\tilde{X}''_k$ are their respective k-fold cyclic coverings, then

$$H_1(\tilde{X}_k) \cong Z \oplus A' \oplus A''$$

where

* see Fox's Quick Trip for a longer list